THE FIXED-POINT QUESTION FOR BOUNDED NON-SEPARATING PLANE CONTINUA

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NON-SEPARATING PLANE CONTINUA

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PREFACE

Mathematics is more than a discipline. It is a living and developing art intertwined with the lives of people. It is my hope that Chapter II of this thesis preserves and exposes this exciting aspect of the fixed-point question in such a way that it is accessible to both graduate students in mathematics and to teachers of undergraduate mathematics.

As an art, mathematics has certain fundamental techniques. In particular, the area of fixed-point theory has its specific artistry. Chapter III is my effort to expose the basic techniques in fixed-point theorems.

While writing this thesis, it has been my pleasure to meet some of the great artists in fixed-point theory. I want to express to them my appreciation for any time or interest given to this thesis.

I wish to express my particular appreciation to my major adviser, Dr. John Jobe, who shared with me his view of and his artistry in mathematics, enabling me to successfully complete this thesis. Especially I thank him for his friendship which has encouraged and supported my personal and mathematical development.

I also owe thanks to Dr. James Choike, Dr. Douglas Aichele, and Dr. Ronald McNew. As members of my committee, they provided me with just the right mixture of challenge and encouragement.

To the members of my family, to my friends, and to my religious community, The Adorers of the Blood of Christ, I also express my thanks. In particular I thank Sister Remigia Kerschen for her continuing support and Sister Dolores Strunk for her typing of the final copy of the thesis.

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CHAPTER I

INTRODUCTION

Topic of the Study

The topic of this thesis is a question which has intrigued and baffled mathematicians for decades: "Does an arbitrary bounded plane continuum which does not separate the plane have the fixed-point property?" Outstanding mathematicians have spent years trying to definitively answer this question. Through their efforts they have proved many directly related theorems, as well as partial solutions to the original problem. It is the purpose of this paper to gather together these results, to present them along with interesting comments in chronological order, and to analyse the techniques used in their proofs.

As a basis for understanding the topic under discussion the terms bounded, continuum, separate the plane, and fixed-point property will be defined.

Definition 1.1. A subset M of the plane is said to be bounded with respect to the metric ρ if and only if $M = \emptyset$ or there exists a real number r such that $\rho(x,y) \leq r$ for all x,y in M.

In this thesis all sets will be situated in the plane unless otherwise specified. The ρ of Definition 1.1 will be the usual metric in the plane.

Before defining continuum, it is necessary to present several preliminary definitions.

Definition 1.2. A point p is a limit point of a set M if and only if every open set containing p contains a point $q \in M$, $q \neq p$.

In the plane countable compactness and compactness are equivalent. Consequently, the following definition will be used for the definition of compactness.

Definition 1.3. A point set M is said to be compact if and only if every infinite subset of M has at least one limit point in M.

Definition 1.4. Two subsets A and B of a space S are said to be separated, denoted sep., if and only if $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$.

Definition 1.5. A subset M of a space S is connected if and only if M is not the union of two separated sets.

Using the concepts of Definitions 1.2-1.5, a continuum can be defined in the following concise terms.

Definition 1.6. A continuum is a compact connected space.

Let the continuum of the question "Does an arbitrary bounded plane continuum which does not separate the plane have the fixed-point property?" be denoted by M. The set M is given to be bounded. However, by the Heine-Borel Theorem in 2-space, it is also closed. Other theorems from Hall and Spencer (18) and properties of the plane, imply M is also 1) T_1, 2) T_2, or Hausdorff, 3) 1st countable, 4) 2nd countable, 5) regular, 6) normal, 7) metric, and 8) separable. The definitions of these properties can be found in Hall and Spencer (18) or any elementary topology book.

A special property of M is that it does not separate the plane. An abundant supply of examples of such continua will be given in Chapter II. It is necessary at this time to define this characteristic.

Definition 1.7. The point set M is said to separate the plane P if and only if $M \subset P$ and $M - P = A \cup B$ sep.

From Bing (5) any plane continuum which does not separate the plane is the intersection of a decreasing sequence of topological disks. That is, M is the intersection of a decreasing sequence of sets homeomorphic to the set $\{(x,y)/\ x^2 + y^2 \leq 1\}$. Definition 1.8 will clarify the meaning of homeomorphic.

Definition 1.8. Let S and T be spaces. Then S is said to be homeomorphic to T if and only if there exists a one-to-one open continuous mapping of S onto T. The mapping is called a homeomorphism.

The question whether or not M has the fixed-point property is a question regarding the behavior of continuous functions $f : M \to M$. Before stating the fixed-point property, continuous will be defined.

Definition 1.9. Let S and T be spaces and $f: S \to T$ a function. Then f is said to be continuous at the point $s \in S$ if and only if, given any open subset G of T such that $s \in f^{-1}(G)$, there exists an open set V of S such that $s \in V \subset f^{-1}(G)$.

It is now possible to define the pivotal concept of this thesis, the fixed-point property.

Definition 1.10. A topological space S is said to have the fixed-point property (fpp) if, given any continuous function f from S into S, there exists a point p such that $f(p) = p$.

During the remainder of this thesis, the fixed-point question or the fixed-point problem will refer to the question asked at the beginning of the chapter: "Does an arbitrary bounded plane continuum which does not separate the plane have the fixed-point property?". As was mentioned earlier, the setting will be the plane unless otherwise stated. One basic and famous fixed-point result will be assumed, namely, the Brouwer Fixed-Point Theorem.

Theorem 1.1. (Brouwer Fixed-Point Theorem) Let $I^n = \{(x_1, x_2, \ldots, x_n) / 0 \leq x_i \leq 1, \text{ for } i = 1, 2, \ldots, n\}$. Let $f\colon I^n \to I^n$ be continuous. There is a point z in I^n such that $f(z) = z$.

Procedure

Chapter II contains the important dates and theorems in the study of the fixed-point question. These are arranged in chronological order and interspersed with both examples and related comments. The purpose of this chapter is to identify and correlate important fixed-point results.

All major theorems stated in Chapter II except those of Harold Bell and Charles Hagopian have their proofs analyzed in Chapter III. Different titles are attached to different techniques of proof. Common and distinguishing elements are discussed and illustrated. Short proofs are presented in their entirety. Longer proofs are presented in shortened form. Their complete proofs are located in Appendix A. All proofs are indented and single-spaced, as a handy reference for the reader and as a means of emphasizing the techniques. Lemmas used in the proofs were not proved in detail in the published papers. They were proved by the author of this thesis, but are not included in the text because of length.

Chapter IV is dedicated to the exposition of Bell's paper and dependent results. Proofs in this chapter are presented in their entirety.

Preliminary Definitions

In this section definitions will be stated that are needed in the thesis and which may not be immediately familiar to the reader.

Definition 1.11. If $f(z)$ is defined in a finite domain G, and is differentiable in z at each point of G, then $f(z)$ is said to be an analytic function in G.

Definition 1.12. An arc is a set homeomorphic to the unit interval, or equivalently, an arc is any compact nondegenerate continuum that has exactly two non-cut points.

Definition 1.13. The point set M is said to be arcwise connected if every two points of M are the endpoints of an arc in M.

Definition 1.14. The boundary B of a point set M in the space S is a set such that $b \in B$ if and only if for every open set U such that $b \in U$ there exists an $x \in M$ and $y \in S - M$ such that $\{x,y\} \subset U$.

Definition 1.15. A maximal connected subset of a point set M is called a component of M.

Definition 1.16. If M is a connected set and p is a point of M such that the set $M - p$ is not connected, then p will be called a cut point of M.

Definition 1.17. Let A be a subset of a metric space S with metric ρ. The diameter, $\delta(A)$, is defined by 1) $\delta(A) = 0$ if $A = \emptyset$, 2) $\delta(A)$ is the least upper bound of $\rho(x,y)$ for all $x,y \in A$ if $A \neq \emptyset$ and A is bounded, and 3) $\delta(A) = \infty$ if A is not bounded.

Definition 1.18. A space S is said to be locally connected at a point p if and only if, given any neighborhood U of p, there exists a connected neighborhood V of p such that $V \subset U$.

Definition 1.19. If H and K are two mutually exclusive closed point sets, the continuum M is said to be an irreducible continuum from H to K if M intersects both H and K but no proper subcontinuum of M intersects both of them.

CHAPTER II

HISTORY OF THE FIXED-POINT QUESTION

The literature of the five decades from 1925 to 1975 contains various theorems and comments regarding the question whether or not an arbitrary plane continuum which does not separate the plane has the fpp. It is the purpose of this chapter to present these ideas in chronological order, showing their interrelationships.

Early Years: 1925-1937

The earliest known printed reference to this problem is found in W. L. Ayres' (2) article "Some Generalizations of the Scherrer Fixed Point Theorem" which appeared in the _Fundamenta Mathematicae_ in 1930. In this article Ayres refers to the problem as a "well-known problem."

In a brief discussion with the author of this thesis on March 15, 1974, K. Kuratowski stated that so far as he knows Ayres was the first to mention this fixed-point problem in print. He said the problem was much discussed in Poland before World War I, however, and, somewhat jokingly, suggested one might call it a "Polish problem."

Since Ayres was in Warsaw at the time that he wrote his article, it seems almost certain that he heard the question discussed there and, consequently, considered it to be well-known. Although his article is the first printed reference to the problem, his main theorem is a generalization of an earlier theorem by W. Scherrer (40). Consequently, the historical sketch will begin with Scherrer's (40) article "Uber ungeschlossene stetige Kurven" which was published in 1926. In that article, Scherrer proved the following theorem:

Theorem 2.1. A homeomorphic mapping of a dendrite into itself has at least one fixed-point.

In considering the meaning of this theorem relative to the fixed-point problem the following definitions are needed:

Definition 2.1. A simple closed curve is a nondegenerate continuum which is disconnected by the omission of any two of its points.

Definition 2.2. A locally connected continuum M is said to be a dendrite provided it contains no simple closed curve, or equivalently, provided a unique arc exists between any two points.

In Whyburn (31) page 107, it is proven that a dendrite does not separate the plane. Consequently, if functions are restricted to homeomorphisms, Scherrer's theorem answers the restricted fixed-point question for the class of plane continua which are locally connected and contain no simple closed curves. Examples of such curves are given in Examples 1 and 2.

Example 1. Let D be the union of three arcs ab, ac, and ad having pair-wise only the point a in common. This is called a triod.

Example 2. Let D consist of the unit interval on the x-axis and the vertical segments defined by $0 \leq y \leq 1/2^n$ and $x = (2k - 1)/2^n$, where $1 \leq k \leq 2^{n-1}$ for $n = 1,2,\ldots$. See Figure 1.

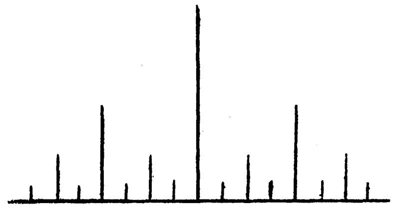


Figure 1. Dendrite

Scherrer's proof that the homeomorphic mapping of a dendrite into itself has a fixed-point is based on the characterization of a dendrite that between any two points there exists a unique arc. It makes frequent use of the fact that f is a one-to-one mapping. The major technique employed is the "dog-chases-rabbit" technique which will be discussed in Chapter III.

In 1929 Fundamenta Mathematicae contained an article by K. Kuratowski (29) entitled "Sur quelques theorems fundamentaux de l'Analysis situs." This article, like Scherrer's, makes no specific mention of the fixed-point question, but it contains a theorem which provides a necessary condition for a locally connected continuum to possess the fpp. That theorem is stated here as Theorem 2.2.

Theorem 2.2. In order that a locally connected continuum have the fpp it is necessary that it be unicoherent.

The concept of unicoherence is a useful property and may be defined as follows:

Definition 2.3. A continuum M is said to be unicoherent provided that however M be expressed as the union of two continua H and K, the set $H \cap K$ is a continuum.

It is interesting to note that while a circle is not unicoherent, a circle with a spiral around it as given in Example 3 is unicoherent.

<u>Example 3.</u> Let $M = \{(x,y)/\ x^2 + y^2 = 1\} \cup \{(1 + t)\cos \pi/t,$ $(1 + t)\sin \pi/t/\ 0 < t \leq 1\}$. See Figure 2.

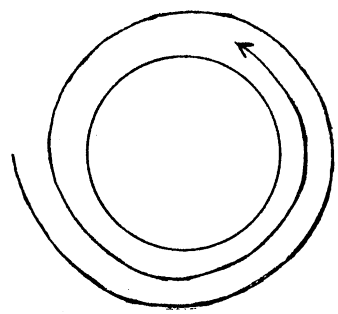


Figure 2. Unicoherent Continuum not Locally Connected

Examples 1 and 2 are examples of locally connected unicoherent continua since each dendrite is a locally connected continua and, according to Whyburn (50) page 88, is unicoherent.

Kuratowski's proof of Theorem 2.2, which is classified as an "immediate" proof, develops a continuous function without a fixed-point. The development of this function is possible because the continuum is locally connected. In fact, without local connectivity unicoherence is not necessary for the fpp. This is illustrated in Example 4 where the continuum M has the fpp but is not unicoherent.

Example 4. Let $M = A \cup B$ where $B = \{(0,y)/\ -2 \leq y \leq 1\} \cup \{(x,-2)/\ 0 \leq x \leq 1\} \cup \{(1,y)/\ -2 \leq y \leq 0\}$ and $A = \{(x,y)/\ y = \sin \pi/x,\ 0 < x \leq 1\}$.

When speaking of a locally connected continuum in the plane, the term Peano continuum may be used. In order to clarify the term, it will now be defined.

Definition 2.4. A Peano continuum is a continuous image of $[0,1]$, or equivalently, any connected, locally connected, compact, metrizable space.

Examples 1 and 2 are Peano continua whereas Examples 3 and 4 are not.

Looking at Definition 2.4 and Theorem 2.2, one sees that unicoherence is necessary if a Peano continuum is to have the fpp. Definitions 2.2. and 2.4 indicate that each dendrite is a Peano continuum. However, each Peano continuum is not a dendrite as is illustrated by a disk.

Because of the relationship between dendrites and Peano continua, Ayres' Theorem 2.3 is a generalization of Scherrer's Theorem 2.1.

Theorem 2.3. If the Peano continuum M lies in a plane and does not separate the plane, then every homeomorphism of M into a subset of itself has a fixed point.

This theorem which was published in 1930 applies to nonseparating plane continua containing topological disks whereas Scherrer's Theorem 2.1 tells nothing about such continua.

The article containing Theorem 2.3 also contained a note from the editors saying that due to work by Borsuk "homeomorphism" may be replaced by "continuous transformation." The resulting theorem was published by K. Borsuk (8) in 1932 in "Einige Satze uber stetige Streckenbilder", <u>Fundamenta Mathematicae</u>, and reads as follows:

<u>Theorem 2.4.</u> The plane Peano continua which do not separate the plane are characterized by the fpp.

This theorem is a culmination of the theorems of Scherrer, Ayres, and Kuratowski. It definitively answers the fixed-point question for Peano continua. Other proofs of Theorem 2.4 have been given by Noebeling (38) and Hopf (25).

In Borsuk's proof of Theorem 2.4, as in Ayres' proof of Theorem 2.3, cyclic element theory plays a major role. This theory is discussed relative to these proofs in Chapter III.

As a result of Borsuk's Theorem 2.4 and Kuratowski's Theorem 2.2, it can be argued that any unicoherent continuum M without the fpp is not a Peano continuum. For if M were a Peano continuum it must either separate the plane or not separate it. If it does not separate the plane, Theorem 2.4 implies M has the fpp. This is a contradiction. Therefore, M must separate the plane. By Whyburn (51) page 188, any Peano continuum which separates the plane is not unicoherent. Consequently, M cannot be a Peano continuum.

An example of a unicoherent continuum without the fpp is given in Example 3. A continuous function defined on M which does not have a fixed-point is the function which rotates each point through a positive angle of 30°.

The Years 1938-1950

After Borsuk's proof of the fpp for Peano continua which do not separate the plane, the next prominent contribution toward a solution to the fixed-point problem was made by O. H. Hamilton (22) in 1938. In an article entitled "Fixed Points under Transformations of Continua which are not Connected im Kleinen," Hamilton proved two fixed-point theorems in the plane. The latter of these was generalized in 1967 to become one of the outstanding fixed-point theorems. This generalization was done almost simultaneously in two different ways by H. Bell (3) and K. Sierklucki (41).

The definitions needed for an understanding of Hamilton's theorems are indicative of the changing approach toward the fixed-point question.

Definition 2.5. A continuum is said to be indecomposable provided it is nondegenerate and is not the union of two continua both distinct from it.

Definition 2.6. A domain (open set) D is said to be simply connected if and only if it is connected and contains one of the complementary domains of every simple closed curve that lies wholly in it.

In Figure 3 (a) is pictured a simply connected domain with a decomposable boundary. Figure 3 (b) pictures a domain which is not simply connected but has a decomposable boundary. Example 5 is an example of an indecomposable continuum which is illustrated in Figure 4.

(a) (b)

Figure 3. Decomposable Boundaries

Example 5. Let $C = \bigcap_{i=1}^{\infty} \left(\bigcup_{n=0}^{(3^k-1)/2} [\, 2n/3^k,\ (2n+1)/3^k \,] \right)$. Let M_0 be the union of all semicircles in the upper half plane with both end points being elements of C and center at $(\frac{1}{2},0)$. Let M_i be the union of all semicircles in the lower half plane with both end points being elements of C and center at the point $(5/2.3^i,0)$. Then $M = \bigcup_{i=0}^{\infty} M_i$ is an indecomposable continuum (30).

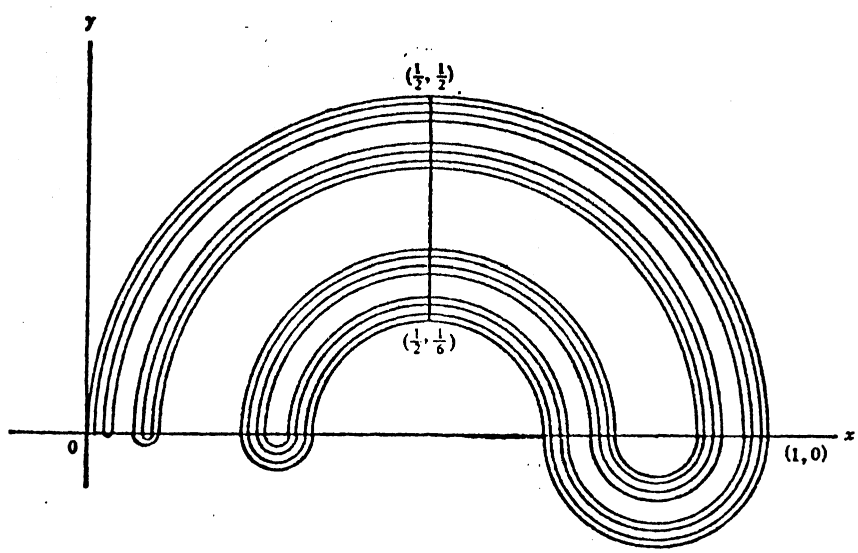


Figure 4. Indecomposable Continuum

Hamilton's two theorems which use the ideas indecomposable and simply connected are presented here as Theorems 2.5 and 2.6.

<u>Theorem 2.5.</u> If M is a compact continuum in the plane which contains no indecomposable continuum, which does not separate the plane and which contains no domain, then every homeomorphism of M into a subset of itself leaves some point invariant.

<u>Theorem 2.6.</u> If D is a bounded simply connected domain in the plane which, together with its boundary, does not separate the plane and whose outer boundary M contains no indecomposable continuum, then every homeomorphism of $\bar{D}$ into itself leaves some point of $\bar{D}$ invariant.

The term outer boundary used in Theorem 2.6 is defined in Definition 2.7.

<u>Definition 2.7.</u> If D is a connected domain and E is a component of the plane minus $\bar{D}$ then the boundary of E will be called the <u>outer boundary</u> of D with respect to E. Any boundary points that are not part of the outer boundary will be considered inner boundary points.

Because of the later generalization of Theorem 2.6, by Bell (3) and Sierklucki (41), Theorem 2.6 is the main interest. Its proof is an "immediate" type proof. In such a proof once a function is defined the solution is easily observed. The function which Hamilton defines involves the clever use of mappings from complex analysis.

Gail Young (53) wrote an article "The Introduction of Local Connectivity by Change of Topology" which appeared in <u>American Journal of Mathematics</u> in 1946. In this article he proved Theorem 2.7, the only result from the 1940's that will be discussed in detail in Chapter III.

<u>Theorem 2.7.</u> Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fpp.

Theorem 2.7 does not require M to be a continuum. However, if M is compact, the conclusion implies M has the fpp. There is an interesting relationship between Theorem 2.7 and results regarding arcwise connected continua of K. Borsuk (7) and W. Holsztyński (24) in 1954 and 1969, respectively. Consequently, additional comments regarding Young's paper will be found in the discussion of their work.

Young uses a "change of topology" technique to effect the proof of Theorem 2.7. An exposition of the proof is located in Chapter III.

The examples of dendrites given in Examples 1 and 2 satisfy the hypothesis of Theorem 2.7. In addition, however, they are locally connected. Example 6 is a continuum which satisfies the hypothesis of Theorem 2.7 but is not a dendrite since it is not locally connected.

<u>Example 6.</u> Let $C = \{(x,y)/\ 0 \leq x \leq 1,\ y = 0\} \cup \{(x,y)/\ x = 1/2^n$ for $n = 1,2,\ldots,\ 0 \leq y \leq 1\} \cup \{(x,y)/\ x = 0,\ 0 \leq y \leq 1\}$.

The Years 1951-1964

In 1951 O. H. Hamilton (20) proved a theorem which answered the fixed-point question for a group of continua called chainable or snake-like. His theorem appeared in "A Fixed Point Theorem for Pseudo-Arcs and Certain Other Metric Continua" in the <u>Proceedings of the American Mathematical Society</u>. The continua covered by the term chainable include such a simple continuum as an arc, such a complicated continuum as that of Example 5, and such a "snake-like" continuum as the pseudo-arc. The last of these will be described in Example 7. First the terms chain and chainable must be defined.

Definition 2.8. A chain is a finite collection of open sets $d_1, d_2, \ldots, d_n$, called links, such that d_i intersects d_j if and only if $i = j - 1$, j, or $j + 1$. If the elements of the chain are of diameter less than $\varepsilon > 0$, the chain is called an ε-chain.

Definition 2.9. If M is a continuum, then M is a chainable continuum if and only if for every real number $\varepsilon > 0$, there exists an ε-chain covering M.

A triod, Example 1, is a continuum which is not chainable. For a chainable continuum, it is possible, and even desirable to define an order on each chain C.

Definition 2.10. A link d_i of a chain C will be said to precede the link d_j in C if $i < j$. The link d_i follows d_j in C if $i > j$.

Hamilton's explicit use of this order and the ε-chains for every $\varepsilon > 0$ enabled Hamilton to prove Theorem 2.8 using a variation of the "dog-chases-rabbit" type proof.

Theorem 2.8. Let D_1, D_2, $D_3, \ldots$ be a sequence of chains such that
i) $\overline{D}_1$ is a compact nonvacuous metric space,
ii) $\overline{D}_{i+1}$ is a subset of $\overline{D}_i$ for each i,
iii) $\lim_{i \to \infty} \triangle(D_i) = 0$ where $\triangle D_i$ signifies the maximum diameter of a link of a chain D_i.
Let M designate the continuum which is the intersection of the $\overline{D}_i$.
Then if f is a continuous transformation of M into a subset of itself,

there exists a point p of M such that $f(p) = p$.

The pseudo-arc is an interesting special case of the class of continua described in the hypothesis of Theorem 2.8. It is also an example of a continuum which was not known to have the fpp by any theorem previous to Theorem 2.8. The pseudo-arc will now be defined and illustrated for the first two steps with the first chain containing five links. T. McKellips (32) contains a detailed study of the pseudo-arc.

Example 7. A pseudo-arc joining two points a, b in the plane is any set in R^2 resulting from the following construction. Let $\{D_i\}$ be a sequence of chains such that

i) the diameter of each open set in D_i is less than $1/i$,

ii) the closure of each link of D_{i+1} is contained in some link of D_i,

iii) D_{i+1} is crooked in D_i, that is if d_m^{i+1}, d_n^{i+1} are in D_{i+1} with $m<n$ and $d_m^{i+1} \subset d_h^i$, $d_n^{i+1} \subset d_k^i$ with $|k - h| > 2$ then there exist d_t^{i+1}, d_s^{i+1} in D_{i+1} with $m < s < t < n$ such that D_s^{i+1} is contained in a link of D_i adjacent to d_k^i and similarly d_t^{i+1} is contained in a link adjacent to d_h^i,

iv) a is in the first link of each D_i and b is in the final link of each chain.

If $D_i^* = \bigcup_{k=1}^{\infty} D_k^i$ denotes the set of all elements of D_i then,

$X = \bigcap_{i=1}^{\infty} D_i^*$ is a pseudo-arc.

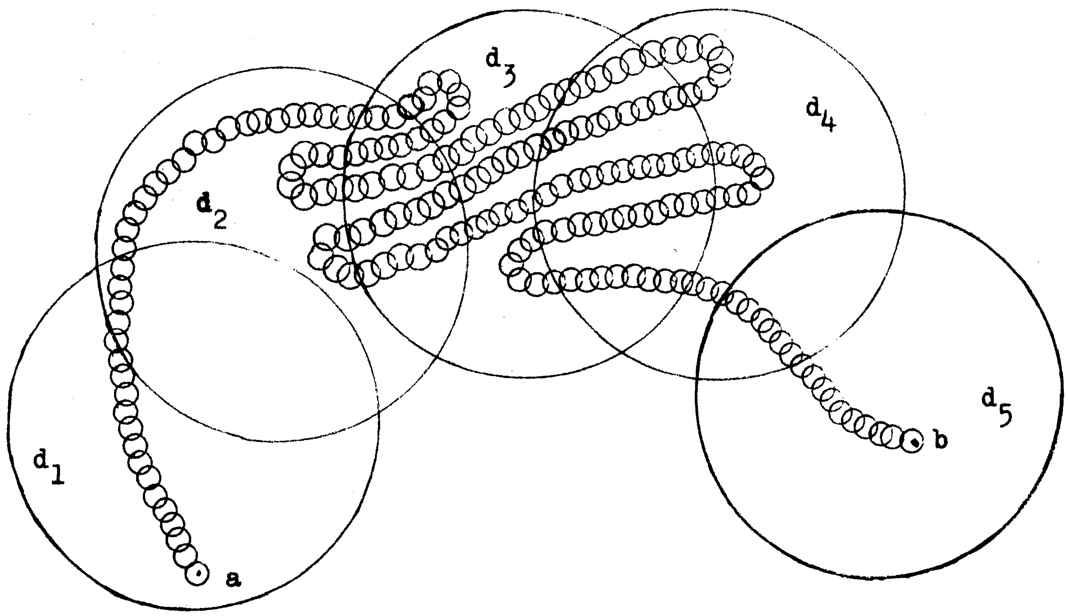


Figure 5. Pseudo-Arc

Another proof of Theorem 2.8 was given by M. Stanko (43) in 1964. In 1965, E. Dyer (14) extended Theorem 2.8 by proving that the Cartesian product of any collection of compact chainable continua has the fpp.

The same year that Hamilton's Theorem 2.8 appeared in print, M. L. Cartwright and J. E. Littlewood (10) published a theorem dealing with continuous orientation preserving functions.

Definition 2.11. Let x,y, and z be three points situated counterclockwise from x to z on the simple closed curve J. Then the continuous function f: $J \to J_i = f(J)$ is said to be orientation preserving provided f(x), f(y), and f(z) are situated counterclockwise in J_i.

The rotation of a circle through an angle θ, $0 \leq \theta \leq 2\pi$ is orientation preserving. The 180° revolution of the unit circle about the x-axis is not orientation preserving.

Cartwright and Littlewood's (10) theorem which appeared in "Some Fixed-Point Theorems" in Annals of Mathematics, reads as follows:

Theorem 2.9. If f is a one-to-one continuous and orientation preserving transformation of the Euclidean plane P onto itself which leaves a bounded continuum M invariant and if M does not separate P then some point of M is left fixed by f.

Cartwright and Littlewood's proof of Theorem 2.9 is based on prime end theory and requires the proof of 32 lemmas and four theorems. In 1954 Hamilton (21) in "A Short Proof of the Cartwright-Littlewood Fixed Point Theorem" in the Canadian Journal of Mathematics, proved Theorem 2.9 without considering prime ends. His proof is based on one stated lemma. However, it also involves the use of a large number of unstated theorems regarding the plane. Since Hamilton's proof uses the techniques of point set topology, we shall consider it rather than Cartwright and Littlewood's proof.

The bulk of Hamilton's proof is the proof of a lemma originally proved by M. H. A. Newman in an unpublished paper. This lemma provides a new function based on the original function of the theorem. After this function is obtained, the proof of the theorem is easily observed. Consequently, the proof is classified as an "immediate" proof.

Also appearing in 1954 in Bulletin De L'Academie Polonaise Des Sciences was an article by Borsuk (7) entitled "A Theorem on Fixed Points." In this article he proved Theorem 2.10 which is closely related to Hamilton's Theorem 2.5. Borsuk's theorem requires the definition of one-dimensional continuum.

Definition 2.12. A continuum M is said to be of dimension one provided each point of M is contained in arbitrarily small neighborhoods (relative to M) whose boundaries are totally disconnected.

Example 1 is a continuum of one-dimension. The closure of Figure 3 (a) is a continuum which is not one-dimensional.

Borsuk's Theorem 2.10 is

<u>Theorem 2.10.</u> Let A be an arcwise connected hereditarily unicoherent one-dimensional continuum. Every continuous mapping of A into A has a fixed point.

At first reading it may not be obvious that Theorem 2.10 is closely related to Theorem 2.5. By Kuratowski (30) page 207, an indecomposable continuum contains no arc. By elementary arc properties and hereditary unicoherence any subcontinuum of A of Theorem 2.10 is arcwise connected and, consequently, decomposable. This means A contains no indecomposable continuum. From the definition of one-dimensional, the continuum A contains no domain. Thus, if A does not separate the plane, then A satisfies the hypothesis of Theorem 2.5.

Theorem 2.10 is also related to Young's Theorem 2.7. The precise relationship will be explored in the discussion of W. Holsztynski's (24) 1969 paper "Fixed Points of Arcwise Connected Spaces." Borsuk in his 1954 paper made no mention of Young's Theorem 2.7. Young (52), however, published a paper in 1960, "Fixed Point Theorems for Arcwise Connected Continua", in which he used his Theorem 2.7 to obtain an easy proof of Theorem 2.10.

Borsuk's proof of Theorem 2.10 is a variation of the "dog-chases-rabbit" type proof. It may be found in its entirety in Appendix A.

In 1957 in an article entitled "Mobs, Trees, and Fixed Points" which appeared in the Proceedings of the American Mathematical Society, L. E. Ward (48) proved that a generalized tree has a fixed point. In a separable space such as the plane, his result is equivalent to Borsuk's Theorem 2.10.

In 1959 Ward (47) gave an erroneous proof of a fixed-point theorem for a class of arcwise connected nonunicoherent continua. This proof which appeared in "A Fixed Point Theorem for Chained Spaces" in the Pacific Journal of Mathematics was not corrected until 1972. A correct proof was given by Smithson and Ward (42) in "The Fixed Point Property for Arcwise Connected Spaces: A Correction" in the same journal as the original proof. The theorem as it appeared in the corrected article is stated here.

Theorem 2.11. If X is an arcwise connected Hausdorff space which contains no circle and if there exists $e \in X$ such that K_r has the fpp for each e-ray, then X has the fpp.

The theorem does not require compactness or boundedness. However, their addition to the hypothesis does not change the conclusion.

In the statement of Theorem 2.11 two new concepts were introduced, namely, e-ray and K_r. These will now be defined.

Definition 2.13. Let X be an arcwise connected Hausdorff space and $e \in X$. An e-ray in X is the union of a maximal nest of arcs ex.

Definition 2.14. Let X be an arcwise connected Hausdorff space and $e \in X$. If R is an e-ray in the space X and $x \in R$, let $A(R,x)$ be the closure of $(R\text{-}ex) \cup x$. Define $K_r = \cap \{ A(R,x) / x \in R \}$.

In the plane the half-line $[0, \infty)$ is an O-ray. In Example 4, A is an e-ray of M with endpoint $(0,1) = e$. The half-line has $K_r = \emptyset$. The set A of Example 4 has $K_r = \{(0,y)/ -1 \leq y \leq 1\}$. If X is compact, $K_r \neq \emptyset$ since K_r is a nested collection of nonempty closed subsets.

In the proof of Theorem 2.11, Smithson and Ward use a variation of the "dog-chases-rabbit" technique. This technique, or a variation of it, has been used consistently in the fixed point theorems for acyclic continua which are arcwise connected.

Since the continuum of Example 4 satisfies the hypothesis of Theorem 2.11, it has the fpp. However, in the year 1960 there was still no conclusive answer to the fixed-point question for arbitrary arcwise connected continua which contain no circle. In Young (52) page 884, is an example of an arcwise connected continuum which contains no simple closed curve and that does not have the fpp. This continuum is not in the plane. It is presented here because it is part of the mathematical history of the question under consideration and because it may have influenced mathematicians in the 60's to conjecture that there existed an arcwise connected plane continuum without the fpp. This is interesting in light of C. Hagopian's (16) proof in 1971 that every bounded arcwise connected plane continuum which does not separate the plane has the fpp.

Example 8. The continuum $M = C_1 \cup C_2 \cup L_1 \cup L_2 \cup R$ where C_1, C_2, L_1, L_2, and R are defined as follows: C_1 is a continuum which is located in the lower xy-plane, which is homeomorphic to the closure of the graph $y = \sin 1/x$, $0 < x \leq \Pi$, and which joins the point $(2,0,0)$ to the interval $[-1, -3]$ of the x-axis having $[-1, -3]$ corresponding to the limiting interval of the graph: C_2 is the image of C_1 under a 180°

rotation of the xy-plane about the origin; L_1 and L_2 are straight line intervals joining (0,0,1) to (2,0,0) and (-2,0,0), respectively; R is a set of points homeomorphic to a half-open interval and which 1) has only (0,0,1) in common with $C_1 \cup C_2 \cup L_1 \cup L_2$, 2) "spirals down" to $C_1 \cup C_2$ in such a way that a) there is a sequence of arcs $\{X_i\}$, $R = \cup X_i$, with $X_i \cap X_j = \emptyset$ for $|j-i| \neq 1$, $X_i \cap X_j$ an endpoint for $|j-i| = 1$, and b) $C_i = \lim_{j\to\infty} X_{2j}$, $C_2 = \lim_{j\to\infty} X_{2j+1}$.

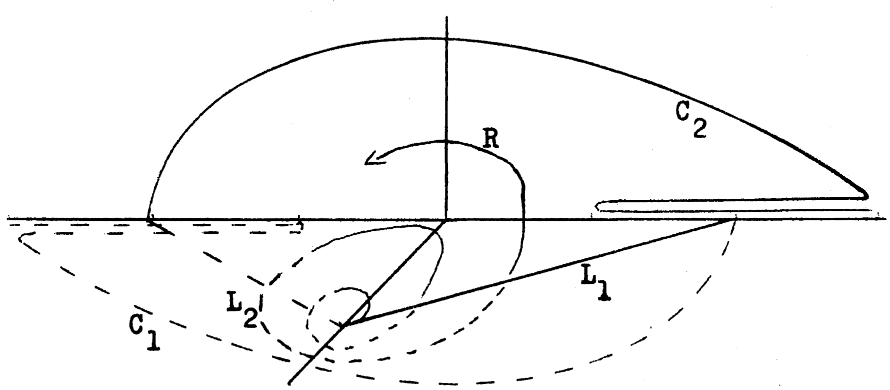


Figure 6. Arowise Connected Continuum without Fixed-Point

This continuum does not satisfy the hypothesis of Theorem 2.11. Let e = (0,0,1). Then R is an e-ray with $K_r = C_1 \cup C_2$. Let f be a 180° rotation of $C_1 \cup C_2$ about the origin. Then f is a continuous function, f: $K_r \to K_r$ and K_r has no fixed-point. In the article in which Example 6 appeared, Young (52) page 884, stated "I have no such example (arowise connected continuum without fpp) in the plane..." He did not conjecture as to whether or not such an example existed.

The Years 1965-1973

In 1965 Harold Bell (3) submitted to the editors of the Transactions of the American Mathematical Society an article entitled "On Fixed Point Properties of Plane Continua." A revised form of the article was submitted in 1966, and the article actually appeared in print in 1967. In the article Bell proved one of the outstanding known results regarding the fpp. The importance of Bell's paper will be discussed in greater detail after the necessary terminology has been introduced. His paper is not easily read because it introduces many specially defined sets. The proof, in fact, baffles many topologists. As a result, the article is often questioned in conversation. No one, however, has challenged the result in print.

In 1968 an article by K. Sieklucki (41) entitled "On a Class of Plane Acyclic Continua with the Fixed Point Property" appeared in Fundamenta Mathematicae. In this article, Sieklucki proved a result equivalent to that of Bell. Since then topologists have proceeded to use Bell's theorem to obtain other results. An outstanding example of this is Hagopian's proof that every arcwise connected plane continuum which does not separate the plane has the fpp.

In order to understand the statement of Bell's theorem, the definition of $T(M)$ is needed.

Definition 2.15. Let M be a bounded set of the plane. The set $T(M)$ is the smallest bounded subset of the plane that contains M and does not separate the plane, or equivalently, $T(M)$ is the complement of the unbounded component of $P - M$ where P is the plane.

If M is a bounded plane continuum not separating the plane, $T(M) = M$. Using this equality Theorem 2.12, a result justified by Bell, can be stated.

Theorem 2.12. Let $f: M \to M$ be a continuous function defined on a plane continuum which does not separate the plane. Then f either fixes a point or there exists an indecomposable continuum Q contained in M such that $Q = f(Q)$.

Instead of proving Theorem 2.12, Bell proved a more general result involving locally bounded functions. The definition of such a function will now be given.

Definition 2.16. A function defined on a set A will be said to be locally bounded at $x \in \bar{A}$ if there exists an open set U containing x for which $f(U \cap A)$ is bounded. The function f is said to be locally bounded if it is locally bounded at each point of $\bar{A}$.

Even though the word function is used in Definition 2.16, the function f may be multi-valued. The step-function $f: R \to R$ with $f([n, n+1)) = n$ is an example of an unbounded, non-continuous, locally bounded function.

The set $f_c(a)$ is also used in Bell's generalized theorem. For the step function just described and a the point (1,0), the set $f_c(a)$ is the line segment from (1,0) to (1,1).

Definition 2.17. Let f be a locally bounded function defined on the set A, $a \in \bar{A}$, and $C(f(U \cap A))$ be the intersection of all convex sets containing $f(U \cap A)$. The set $f_c(a) = \{ [C(f(U \cap A))] / a \in U$ and U is open $\}$. If $B \subset \bar{A}$, then $f_c(B)$ will denote $\cup \{ f_c(x)/ x \in B \}$.

Theorem 2.13 is Bell's generalized theorem as he stated it.

<u>Theorem 2.13.</u> Let M be a continuum and let g be a locally bounded function defined on the plane for which $g(x) \in T(M)$ if x is not in the interior of $T(M)$. If g is continuous at each point of M then either g_c has a fixed point in $T(M)$ or there is an indecomposable continuum $Q \subset M$ such that $Q = g(Q)$.

Theorem 2.13 will now be stated in terms of a bounded plane continuum which does not separate the plane.

<u>Theorem 2.14.</u> Let M be a bounded plane continuum which does not separate the plane, and let g be a locally bounded function defined on the plane such that $g(x) \in M$ if x is not in the interior of M. If g is continuous at each point of M, then either g_c has a fixed-point in M or there is an indecomposable continuum $Q \subset M$ such that $Q = g(Q)$.

Any continuous function mapping M into M can be extended to a locally bounded function on the entire plane such that $g(x) \in M$. Such an extension will be called an lb extension and is defined in Definition 2.18.

<u>Definition 2.18.</u> Let f be a continuous function mapping the bounded plane continuum Q into itself. The multi-valued function g defined by

$$g(x) = \begin{cases} f(x) \text{ if } x \text{ is in } Q \\ \\ f(m) \text{ for some } m \in T(Q) \text{ for which } \; |x-m| = \inf \{ |x-y| / y \in T(Q) \} \end{cases}$$

is a locally bounded multivalued function and will be called an <u>lb extension</u> of f.

Letting the g in Theorem 2.14 be an lb extension of a continuous function f: M $\Rightarrow$ M where M is a bounded plane continuum which does not separate the plane, justifies Theorem 2.12. The importance of this theorem among the fixed-point results can hardly be exaggerated. It generalizes Hamilton's Theorems 2.5 and 2.6. Since by Borsuk (7) page 17 an arcwise connected hereditarily unicoherent continuum does not contain any indecomposable continuum, Borsuk's Theorem 2.10 is a special case of Bell's theorem. Bell's result provides the powerful tool that enables Hagopian to prove that every bounded arcwise connected plane continuum which does not separate the plane has the fpp. Hagopian's theorem, in turn, encompasses the results on Peano continua. In other words, it generalizes the work of Scherrer, Ayres, and Borsuk as presented in Theorems 2.1, 2.3, and 2.4. Hagopian's theorem also encompasses the results of Smithson and Ward as presented in Theorem 2.11.

In addition, Bell's paper provides a criteria for determining whether or not a bounded plane continuum which does not separate the plane has the fpp. If the continuum has a boundary that is hereditarily decomposable, it has the fpp. If the continuum is known not to have the fpp, its boundary must contain an indecomposable continuum.

Bell's Theorem 2.14 does allow a continuum to be indecomposable and to have the fpp. Such continua do exist since the pseudo-arc which is indecomposable has the fpp by Hamilton's Theorem 2.8.

An exposition of the approach and proof used by Bell in Theorem 2.14 is a major part of Chapter IV. There an attempt is made to improve the readability of Bell's paper by the addition of lemmas, illustrative figures, and more complete explanations.

The same year that Bell's article appeared, an article by Bing (4) entitled "Challenging Conjectures" appeared in the American Mathematical Monthly. In this article Bing mentions how he has received papers to referee which contained erroneous proofs of the fpp, as well as how he has received at least one preprint of an erroneous counterexample. He also tells of a lecture he attended on the fpp which was stopped when a question from the audience revealed an error.

This article, along with a 1969 article entitled "The Elusive Fixed Point Property" in the same journal and also by Bing (6), added no new knowledge to the fixed-point question. However, these papers do provide an interesting exposition of this tantalizing problem.

As was mentioned earlier in the chapter, a 1969 article by Holsztynski (24) clarifies the relationship between Young's Theorem 2.7 and Borsuk's Theorem 2.10.

Theorem 2.10 holds for arcwise connected hereditarily unicoherent one-dimensional continua, the properties of which enable Borsuk to use Theorem 2.15 as an aid in proving his theorem.

Theorem 2.15. Let A be an arcwise connected and hereditarily unicoherent one-dimensional continuum and f a one-to-one continuous function mapping the ray $[0, \infty)$ into A. Then the closure of the set $P = f([0,\infty))$ is an arc.

While thinking about the statement of Theorem 2.15 it is interesting to ask whether or not its conclusion holds for arcwise connected one-dimensional continua which are not hereditarily unicoherent. This is part of what Holsztynski (24) did in "Fixed Points of Arcwise Connected Spaces." He names such spaces Borsuk-Young or B-spaces.

Definition 2.19. An arcwise connected space X is a B-space if it has the property that for any one-to-one continuous mapping f of the ray $[0, \infty)$ into X, the closure of the set $P = f([0,\infty))$ is an arc.

Using this definition Holsztynski (24) pages 290 and 291, proves that any arcwise connected, hereditarily unicoherent, and one-dimensional continuum is a B-space. That a B-space need not be hereditarily unicoherent is demonstrated by Example 9.

Example 9. Let $p_n = (1, 1+1/n)$, $q_n = (2, 1/n)$, $I = \{(x,y)/\ 0 \leq x \leq 2,\ y = 0\}$, L_n the segment joining $(0,0)$ and p_n, and L'_n, the segment joining p_n and q_n. Then $M = I \cup (\bigcup_{n=1}^{\infty} L_n) \cup (\bigcup_{n=1}^{\infty} L'_n)$.

This space is not unicoherent since $I \cup [(\bigcup_n L_n) \cup (\bigcup_n L'_n)] = I \cup S$ is connected. Whereas, $I \cap S = \{(0,0), (2,0)\}$ is not connected.

Definition 2.20 is shown by Holsztynski to be equivalent to Definition 2.19 in the plane.

Definition 2.20. An arcwise connected space X is a B-space if and only if in the space X the union of every monotone increasing sequence of arcs is contained in an arc.

On the basis of this definition, it is clear that any Hausdorff B-space by definition satisfies the hypothesis of Theorem 2.7 and has the fpp. Since any hereditarily unicoherent one-dimensional continuum is a B-space, Borsuk's Theorem 2.10 is a special case of Young's Theorem 2.7.

Holsztynski's paper contains a more general fixed-point theorem for B-spaces. His theorem when situated in the plane is equivalent to Young's Theorem 2.7.

In 1970, L. Mohler (34) proved a theorem regarding continua which are hereditarily divisible by points. It has been conjectured by Knaster that all such continua have the fpp. Mohler's theorem proves the conjecture to be true for hereditarily unicoherent continua with $\tau(H) \neq \infty$. Such continua are hereditarily divisible by points by Theorem 2.16. Before stating the theorems being discussed here, it is necessary to define hereditarily divisible by points and $\tau(H)$.

Definition 2.20. A continuum M is said to be hereditarily divisible by points if each non-degenerate subcontinuum has a cutpoint.

Dendrites, the closure of the set $\{(x,y)/\ y = \sin 1/x,\ 0 < x \leq \pi\}$ and the Cantor fan of Example 10 are hereditarily divisible by points. A circle is an example of a continuum which is not divisible by points.

Example 10. Let M consist of straight segments joining the points $(\frac{1}{2},\frac{1}{2})$ with all points $(x,0)$ where x is a point of the Cantor set. (See Example 5.)

The notation $\tau(H)$ denotes a degree of non-local connectedness of H which was defined by Charatonik (11) page 190, and generalized by Mohler (34). The concept is easily grasped even though the required number of definitions and theorems seems formidable. The development given here is that given by Mohler.

Theorem 2.16. A continuum H is hereditarily unicoherent if and only if given any set $X \subset H$, $X \neq \emptyset$, there is a unique subcontinuum $I(X)$ of H which is irreducible with respect to containing X, that is, no proper subcontinuum of $I(X)$ contains X.

<u>Definition 2.21.</u> If X is a topological space, then <u>N(X)</u> denotes the points at which X fails to be locally connected.

Examples of N(X) are easily observed. If X is a dendrite, $N(X) = \emptyset$. If X is the closure of the set $\{(x,y)/ \ y = \sin 1/x, \ 0 < x \leq \pi\}$, then $N(X) = \{(0,y)/ \ -1 \leq y \leq 1\}$. Referring to Example 10, N(X) is the Cantor set.

<u>Definition 2.22.</u> If H is a hereditarily unicoherent continuum, then define <u>J(H)</u> = I(N(H)).

For the closure of the set $\{(x,y)/ \ y = \sin 1/x, \ 0 < x \leq \Pi\}$, the set J(H) = N(H). If H is the set of Example 10, J(H) is the entire continuum.

<u>Definition 2.23.</u> Let H be a hereditarily unicoherent continuum. For each ordinal α define $\underline{J^{\alpha}(H)}$ as follows: $J^{0}(H) = H$; if $\alpha \neq 0$ then,

$$J^{\alpha}(H) = \begin{cases} J(J^{\beta}(H)) \text{ if } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} J^{\beta}(H) \text{ if } \alpha = \lim_{\beta<\alpha} \beta \end{cases}$$

<u>Definition 2.24.</u> If H is a hereditarily unicoherent continuum, define $\underline{\tau(H)}$, the degree of non-local connectedness of H, as follows:

$$\tau(H) = \begin{cases} \min\{\alpha / J^{\alpha+1}(H) = \emptyset\} \text{ if } \{\alpha / J^{\alpha+1}(H) = \emptyset\} \neq \emptyset \\ \infty \text{ otherwise} \end{cases}$$

It is easily seen that $\tau(H) = 0$ for dendrites, 1 for the closure of the sin 1/x graph, $0 < x \leq \Pi$, and ∞ for the Cantor fan of Example 10.

Mohler first proves a theorem which gives the relationship among hereditarily unicoherent continua, $\tau(H)$, and hereditarily divisible by points. This theorem is Theorem 2.17.

<u>Theorem 2.17.</u> If H is a hereditarily unicoherent continuum and $\tau(H) \neq \infty$, then H is hereditarily divisible by points.

Using Theorem 2.17 it is possible to see the connection between the theorem of interest, Theorem 2.18, and divisibility by points.

<u>Theorem 2.18.</u> If H is a hereditarily unicoherent continuum and $\tau(H) \neq \infty$, then H has the fixed-point property.

In order to prove Theorem 2.18, Mohler proves a generalized form of Young's Theorem 2.7 which holds in a non-metric space. Instead of arcs, topological chains are used in the statement of the generalization. Since the setting in this thesis is the plane, Young's theorem is sufficient to prove Theorem 2.18.

In the course of proving Theorem 2.18, Mohler proves that the continuum H contains a subset A which satisfies the hypothesis of Theorem 2.7. Since the hypothesis of Theorem 2.7 requires a monotone increasing sequence of arcs, the technique used by Mohler in proving Theorem 2.18 will be called the "sequence of arcs" technique. This method does not directly involve a function and in this way differs from both the "dog-chases-rabbit" and the "immediate" types of proofs.

Since Mohler's proof relies directly on Young's Theorem 2.7, the question arises as to whether or not there exist hereditarily unicoherent continua with $\tau(H) \neq \infty$ that do not have every monotone increasing sequence of arcs contained in an arc. The closure of the set $\{(x,y)/\ y = \sin 1/x,\ 0 < x \leq \pi\}$ is such a continuum.

Charles Hagopian (16) made specific use of Bell's Theorem 2.14 to prove an important fixed-point result in 1971. "A Fixed Point Theorem for Plane Continua" which appeared in the Bulletin of the American Mathematical Society contained his proof of Theorem 2.19.

Theorem 2.19. If M is an arcwise connected bounded plane continuum which does not separate the plane, then M has the fpp.

Previously Young, Borsuk, Smithson and Ward in Theorems 2.7, 2.10, and 2.11, respectively, proved the existence of the fpp for special classes of arcwise connected continua. Using Bell's result, Hagopian found it possible to prove the fpp for all arcwise connected plane continua which do not separate the plane. This was a great step forward as it leaves the question open only for continua which are not arcwise connected. Moreover, it gives hope that there may be a general proof for all such continua.

The proof of Theorem 2.19 is located in Chapter IV because of its dependence on Bell's Theorem 2.15. The original proof by Hagopian (16) on pages 353 and 354, contained an error. Consequently, the proof in Chapter IV is a modified version.

In 1972 Hagopian defined a concept he called λ-connectedness. He also generalized Theorem 2.19 to λ-connected continua.

Definition 2.25. A continuum M is said to be λ-connected if every two points of M can be joined by a hereditarily decomposable continuum in M.

Since an arc is a hereditarily decomposable continuum, an arcwise connected continuum is clearly λ-connected. That the converse is not true is illustrated by the closure of $\{(x,y)/\ y = \sin 1/x,\ 0 < x \leq \pi\}$.

Theorem 2.20 is the generalization of Theorem 2.19 to λ-connected continua.

<u>Theorem 2.20.</u> If M is a λ-connected bounded plane continuum that does not separate the plane, then M has the fixed-point property.

The proof of Theorem 2.20 mimics the proof of Theorem 2.19 with the notion of arc replaced by hereditarily decomposable continuum. The paper by Hagopian (17) "Another Fixed Point Theorem for Plane Continua" which contains the proof of Theorem 2.20 also contains a proof of the fact that if M is a bounded plane continuum that does not separate the plane and if $\partial(M)$ does not contain an indecomposable continuum then M is λ-connected. This result is interesting in light of Bell's Theorem 2.14.

Present Efforts

Since 1972, Hagopian has published a number of papers regarding λ-connectedness. It is possible that the development of λ-connectedness for non-separating plane continua will lead to a final answer to the fixed-point question. Hagopian (18) has proven at least one theorem relating λ-connectedness and the fpp for disk-like continua. Such continua are defined in Definition 2.26.

<u>Definition 2.26.</u> A continuum M is <u>disklike</u> if for each $\varepsilon>0$, there exists an ε-map of M onto a disk where an ε-map f is a continuous function defined on M such that for each $q \in f(M)$, the diameter of $f^{-1}(q) < \varepsilon$.

The question "Does every disk-like continuum have the fixed-point property?" was asked by S. Mardesic (31) in 1963. Since nonseparating plane bounded continua are a subset of disk-like continua, an affirmative answer to this question would imply that all nonseparating bounded plane continua have the fpp. Some of the work presently being done regarding disk-like continua may definitively answer the fixed-point question. However, no such result is available at present. In fact some topologists have the feeling a counter-example exists. In a paper entitled "Fixed Point Problems for Disk-Like Continua" which is to appear in the Mathematical Monthly, Hagopian (18) proves a theorem he believes may be helpful to those looking for a disk-like continuum that does not have the fpp. His theorem reads as follows:

Theorem 2.21. Let T be the continuum in E^3 that is the union of the disk $D = \{ (x,y,z) \in E^3 / \, x^2 + y^2 \leq 1 \text{ and } z = 0 \}$ and the arc $A = \{ (x,y,z) \in E^3 / \, x = y = 0 \text{ and } 0 \leq z \leq 1 \}$. If a continuum M has a subcontinuum that is homeomorphic to T, then M is not disk-like.

Certainly many topologists worked to solve the fixed-point problem. Many are continuing this work at the present time. Outstanding among these is O. H. Hamilton. It is evident from Theorems 2.5, 2.6, and 2.8 that he has made major contributions in this area. He is presently working on a proof of the statement: If M is a bounded plane continuum which does not separate the plane and f is a continuous transformation of M onto itself which transforms the boundary B of M into B then for some point p of M, f(p) = p.

The proof of this statement would enable him to prove 1) that every plane continuum which contains no interior and which is the intersection of a monotonic decreasing sequence of topological disks, has the fpp; 2) an indecomposable plane continuum which does not separate the plane has the fpp; and 3) if M is a bounded plane continuum which does not separate the plane and whose boundary Q is an indecomposable continuum and f is a continuous function mapping M onto M and the boundary Q of M onto Q then for some $p \in M$, $f(p) = p$. Using these statements and Bell's Theorem 2.14, he could then prove the comprehensive fixed-point theorem. It is on this note of hopeful expectation that the history of the fixed-point question ends.

CHAPTER III

FUNDAMENTAL TYPES OF PROOFS

In studying proofs of fixed-point theorems one repeatedly encounters certain types of proofs which might be called the basic tools of fixed-point theorists. Some of these are so familiar that they are called by descriptive titles or nick-names. Others are associated with particular mathematicians or particular types of continua. It is the purpose of this chapter to provide not only a compilation of these ideas, but also to describe and analyze them in relation to the theorems in Chapter II in which they were employed.

The Dog-Chases-Rabbit Technique

The first type of proof to be considered is an old friend. It has been around since the 1920's and has been nick-named the "dead-end" or "dog-chases-rabbit" type proof. To illustrate the appropriateness of these names Case (iii) of Scherrer's Theorem 2.1 will be discussed in an intuitive way.

> Theorem 2.1. A homeomorphic mapping of a dendrite into itself has at least one fixed point.

In this theorem f is a homeomorphism defined on the dendrite D. It is assumed that f has no fixed-point. The arcs p_1p_2 and p_2p_3 of D are such that $f(p_1p_2) = p_2p_3$, $f(p_1) = p_2$. In Case (iii) the setting is restricted to p_1p_2 and $p_1p_2 \cap p_2p_3$ is an arc. See Figure 7.

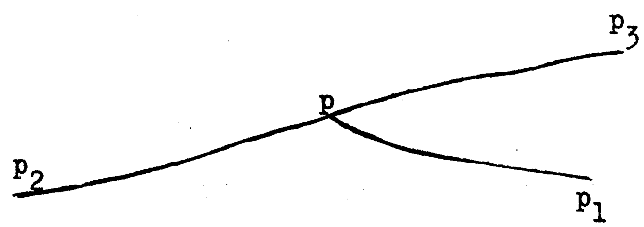


Figure 7. Case (iii) of Theorem 2.1

Let x, the dog, begin at p_1 and move toward p. Since $f(p_1) = p_2$, the rabbit begins at p_2 and moves toward p. Since f is assumed to have no fixed-point, x and f(x) cannot arrive simultaneously at p. Suppose the dog, x, arrives first at p. Then f being restricted to p_1p_2, x, the dog, must move from p toward p_2. As the dog moves toward p_2 and the rabbit moves toward p, the two meet at some point of p_2p.

If the rabbit, f(x) arrives first at p, the "dog-chases-rabbit" proof is ineffective. The failure of the proof is due to the restriction of f to p_1p_2. This prevents the point x, or the dog from moving along pp_3. It is necessary in this subcase to employ the technique of Case (i).

It is only when the dog, x, and the rabbit, f(x), both belong to pp_2 that the "dog-chases-rabbit" technique is effective. A more rigorous presentation of this proof requires a definition of precedes on an arc. It also requires certain basic theorems from Moore (35) which will be stated here without proof.

Definition 3.1. If ab is an arc, the point x is said to *precede* the point y on ab ($x < y$) if x and y are points of ab and either i) y is b, or ii) y separates $\{a, x\}$ from b in ab.

Theorem 3.1. If x and y are two points of the arc ab then either $x < y$ or $y < x$ on ab, and if $x < y$ on ab then y does not precede x on ab.

Theorem 3.2. If ab and bc are arcs such that $ab \cap bc = b$, then $ab \cup bc = ac$.

Theorem 3.3. If ab is an arc and p is a point of ab distinct from a and b, and ab - p is the sum of two separated sets H and K, and H contains a, then $H \cup p$ is the arc ap and H is connected.

It is now possible to consider the proof of Case (iii) of Scherrer's Theorem 2.1. To aid the discussion, the pertinent part of the proof is presented here in shortened form. For the details of the proof see Appendix A.

Theorem 2.1. Case (iii) The set $p_1p_2 \cap p_2p_3$ is an arc pp_2.
Proof:
Suppose f(p) is in pp_2. Inductively define a sequence of points $\{x_n\}$ such that for every n, $x_n < x_{n+1} < f(x_{n+1}) < f(x_n)$ on pf(p). From the induction argument one obtains a sequence of distinct points $\{x_n\}$. The compactness of D implies that some subsequence $\{x_k\}$ of $\{x_n\}$ must converge to a point x_o. Without loss of generality, assume $\{x_n\}$ converges to x_o. The continuity of f implies that $\{f(x_n)\}$ converges to $f(x_o)$.

From the definition of a metric, $\rho(x_o, f(x_o)) = 0$ or $\rho(x_o, f(x_o)) > 0$. If $\rho(x_o, f(x_o)) = 0$, then $x_o = f(x_o)$ and f has a fixed-point. If $\rho(x_o, f(x_o)) > 0$, one is back in Case (i).

The proof depends upon certain properties of arcs as well as the one-to-oneness and the continuity of f. These properties, however, do not characterize the "dog-chases-rabbit" technique.

A sequence of points satisfying the relationships $x_n < x_{n+1} < f(x_{n+1}) < f(x_n)$ was used in the proof. In the analyzation of the proof, it becomes clear that it is the relationships $x_n < x_{n+1}$ and $x_n < f(x_n)$ which convey the idea of a dog chasing a rabbit. To bring the proof to a successful conclusion, it was also necessary for $\rho(x_o, f(x_o))$ to be zero. Otherwise, the approach of Case (i) had to be invoked.

One way of assuring that $\rho(x_o, f(x_o)) = 0$ is to require that for every $\varepsilon > 0$ there exists an n such that $\rho(x_n, f(x_n)) < \varepsilon$. It is this statement along with a sequence of points such that $x_n < x_{n+1}$ and $x_n < f(x_n)$ that characterizes the "dog-chases-rabbit" or "dead-end" type proof. If it is true that for every $\varepsilon > 0$ there exists an x such that $\rho(x, f(x)) < \varepsilon$, but these x values do not necessarily form a sequence with the indicated relationships, then the proof is said to be a variation of the "dog-chases-rabbit" type proof. Hamilton's Theorem 2.8 illustrates such a variation.

> Theorem 2.8. Let $D_1, D_2, D_3, \ldots$ be a sequence of chains such that
> i) $\bar{D}_i$ is a compact nonvacuous metric space,
> ii) $\bar{D}_{i+1}$ is a subset of D_i for each i,
> iii) $\lim \triangle(D_i) = 0$ where $\triangle D_i$ signifies the maximum diameter of a link of the chain D_i.
> Let M designate the continuum which is the intersection of the $\bar{D}_i$. Then if f is a continuous transformation of M into a subset of itself, there exists a point p of M such that $f(p) = p$.

The definition of chain is given in Chapter II as Definition 2.8.

In this proof it is important to note how the order on the links of an ε-chain is used to obtain a point x such that $\rho(x, f(x)) < \varepsilon$. This order and the fact that $\lim \triangle(D_i) = 0$ provide a sequence of points having the property necessary for the "dog-chases-rabbit" proof.

Theorem 2.8.
Proof:
Let $\varepsilon > 0$. There exists a positive integer m such that $\triangle D_m < \varepsilon$. For every p in M, p belongs to some closed link d_i of D_m and f(p) belongs to some closed link preceding d_i, preceded by d_i or equal d_i.

Using the order of the chain, define two non-empty closed sets A and B such that $A \cup B = M$, $A \cap B \neq \emptyset$ and for some q in $A \cap B$, q and f(q) lie together in a closed link d_r of D_m. This forces $\rho(q, f(q))$ to be less than ε. Since ϵ was arbitrary this argument holds for any ϵ and one can define a sequence $\{q_n\}$ having $\rho(q_n, f(q_n)) < 1/n$ for every $n = 1,2,3,\ldots$ By the compactness of M, it can be assumed without loss of generality that $\{q_n\}$ converges to a point q in M.

Thus, for every $\epsilon > 0$ there exists an N such that for every $n > N$, $0 \leq \rho(q, f(q)) \leq \rho(q, q_n) + \rho(q_n, f(q_n)) + \rho(f(q_n), f(q)) < \epsilon$. It follows that $f(q) = q$ and M has a fixed-point.

Another means of obtaining a sequence $\{x_n\}$ such that $\rho(x_n, f(x_n)) < \epsilon$, is to let $x_n = f^n(x_o)$ for some specified x_o. If it can then be shown that $\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0$ and $\{x_n\}$ converges to a unique point, x, it follows that for every $\epsilon > 0$ there exists an N such that for every $n > N$, $\rho(x, f(x)) \leq \rho(x, x_n) + \rho(x_n, f(x_n)) + \rho(f(x_n), f(x)) < \epsilon$ and $f(x) = x$. In the cases where an order is defined and $f(x_n) < f(x_{n+1})$ for every n, this results in the "dog-chases-rabbit" type proof. However, in such cases the term "dead-end" is more descriptive.

If no order is defined but a sequence of points is obtained such that $x_n = f^n(x_o)$, $\lim_{n \to \infty} \delta(x_n, x_{n+1}) = 0$, and $\{x_n\}$ converges to a unique point, the proof is another variation of the "dog-chases-rabbit" proof. An example of such a proof is found in the proof of Case (ii) of Scherrer's Theorem 2.1.

> <u>Theorem 2.1.</u> A homeomorphic mapping of a dendrite into itself has at least one fixed-point.

In Case (ii), the arcs p_1p_2 and p_2p_3 intersect at the point p_2 with $p_2 = f(p_1)$, $p_3 = f(p_2)$. Lemma 1, a compilation of theorems in Whyburn (50), is indispensable in the proof.

<u>Lemma 1.</u> If M is a hereditarily locally connected continuum, then any sequence, $\{R_n\}$, of disjoint connected subsets of M, has the property that $\lim_{n \to \infty} \delta(R_n) = 0$.

> <u>Theorem 2.1.</u> Case (ii) The set $p_1p_2 \cap p_2p_3 = p_2$.
> <u>Proof:</u>
> Because f is a homeomorphism, the set $p_2p_3 \cap p_3p_4 = p_3$ where $f(p_2p_3) = p_3p_4$. Also $p_1p_2 \cap p_3p_4 = \emptyset$. An infinite sequence of arcs p_1p_2, p_2p_3,... can, therefore, be defined such that $p_np_{n+1} = f(p_{n-1}p_n)$, and $p_{n-1}p_n \cap p_np_{n+1} = p_n$. Lemma 1 implies $\lim \delta(p_np_{n+1}) = 0$.
> The choice of p_n implies the sequence $\{p_n\}$ converges to a point p.
> The function f being continuous, implies that $\{f(p_n)\}$ converges to f(p). Therefore, for every $\epsilon > 0$, n can be chosen such that $\rho(p, f(p)) \leq \rho(p, p_n) + \rho(p_n, f(p_n)) + \rho(f(p_n), f(p)) < \epsilon$ and $f(p) = p$.

Another variation of the "dog-chases-rabbit" proof is found in Borsuk's proof of Theorem 2.10.

<u>Theorem 2.10.</u> Every continuous mapping of an arcwise connected and hereditarily unicoherent one-dimensional continuum into itself has a fixed-point.

The proof assumes an arbitrary continuous function f has no fixed-point. This is equivalent to assuming that there exists an $\epsilon > 0$ such that for every point x in the continuum, $\rho(x,f(x)) > \epsilon$. A sequence is then defined inductively from which a contradiction is obtained.

In the proof of Theorem 2.10 the second of Borsuk's lemmas which was stated as Theorem 2.15 will be assumed.

<u>Theorem 2.15.</u> Let A be an arcwise connected and hereditarily unicoherent one-dimensional continuum and f a one-to-one continuous function mapping the ray $[0,\infty)$ into A. Then the closure of the set $P = f([0,\infty))$ is an arc.

The following is a shortened form of Borsuk's proof.

<u>Theorem 2.10.</u>
<u>Proof:</u>

Let A be an arcwise connected, hereditarily unicoherent one-dimensional continuum. Suppose there exists a continuous mapping $f: A \to A$ with no fixed-points. Then there exists an $\epsilon > 0$ such that $\rho(p, f(p)) \geq \epsilon$ for every p in A.

Inductively define a sequence of distinct points $\{a_n\}$ A satisfying the following conditions:

i) $\rho(a_i, a_{i+1}) = \frac{1}{2}\epsilon$ for every $i < n$,

ii) if p is in $a_i a_{i+1}$ then $\rho(a_i, p) < \frac{1}{2}\epsilon$ for $i < n$,

iii) $a_1 a_n = \cup a_i a_{i+1}$,

iv) a_n is in $a_1 f(a_n)$ if $n > 1$.

Having defined such a sequence, let g_n be a homeomorphism mapping $[n-1,n]$ into $a_n a_{n+1}$ in such a manner that $g_n(n-1) = a_n$ and $g_n(n) = a_{n+1}$.

Define $g: [0,\infty) \to P = \cup a_n a_{n+1}$ as follows: $g(x) = g_n(x)$ for $1 \leq x \leq n$. The function g is one-to-one, continuous, and <u>onto</u> P.

By Theorem 2.15, $\bar{P}$ <u>is</u> an arc and there exists a homeomorphism h mapping $\bar{P}$ onto the interval $[0,1]$. The sequence $\{h(a_n)\}$ is monotone and, hence, convergent.

Since h is a homeomorphism, this implies $\{a_n\}$ must also be convergent. This contradicts the fact that $\rho(a_n, a_{n+1}) = \frac{1}{2}\varepsilon$, $n = 1,2,\ldots$ and the proof is complete.

Smithson and Ward's proof of Theorem 2.11 provides another variation of the "dog-chases-rabbit."

> Theorem 2.11. If X is an arcwise connected Hausdorff space which contains no circle and if there exists an e in X such that K_r has the fixed-point property for each e-ray R, then X has the fixed-point property.

An e-ray is defined in Definition 2.13. Order on X is made precise by Definition 3.2.

Definition 3.2. Let X be an arcwise connected Hausdorff space which contains no circle and $e \in X$. Let $x,y \in X$. Define $x \leq y$ if and only if $x \in ey$.

The order of Definition 3.2 is a partial order on X. It may or may not agree with the natural order on an arc. For consider an e-ray R, $x,y \in R$ such that $x < y$ on R. Then $y < x$ on the arc yx.

Theorems 3.4 and 3.5, also from Smithson and Ward (42) pages 512 and 513, are needed for the proof of Theorem 2.11.

Theorem 3.4. Let X be an arcwise connected Hausdorff space which contains no circle and $e \in X$. If $x \in X$ such that $x \not\leq f(x)$ on some e-ray, and if there exists $t \in X$ such that $t \leq f(t) \leq x$ on some e-ray, then f has a fixed-point.

Theorem 3.5. Let X be an arcwise connected Hausdorff space which contains no circle and $e \in X$. If $f\colon X \to X$ is continuous and if p and q are elements of X such that p precedes q in the natural order from f(p) to $f(q)$ on the arc $f(p)f(q)$, then there exists an $x \in pq$ such that $x = f(x)$.

Following a summary of the proof of Theorem 2.11, comments will be made regarding the way in which the proof is a variation of the "dog-chases-rabbit" technique.

Theorem 2.11.

Proof:

Let f be a continuous function $f: X \to X$. Assume $f(e) \neq e$. Let D denote the family of all subsets of X such that $e \in S$, $S \cup f(S)$ is linearly ordered with respect to the partial order of Definition 3.2 and $s \leq f(s)$ for each x in S. By Zorn's lemma, D has a maximal element M.

Suppose $M \cup f(m) \subset ex$ for some $x \in X$. If $x \neq f(x)$ then for some $m \in M$, $m \leq f(m) \leq x$, and f must have a fixed-point by Theorem 3.4.

Assume $x \leq f(x)$ for each x such that $M \cup f(M) \subset ex$. The maximality of M implies $\{x, f(x)\} \subset M$. This means $x \leq f(x)$ and $f(x) \leq x$. Therefore, $x = f(x)$.

If $M \cup f(M) \not\subset ex$ for any $x \in X$ then for some e-ray the following is true: for each $r \in R$ there exists an m called $m(r)$, $m(r) \in M \cup f(M)$ such that $r \leq m(r)$. In other words, $M \cup f(M)$ is cofinal in R. Moreover, M itself is cofinal in R.

If $f(K_r) \subset K_r$ for every K_r, the proof is complete. Assume $f(K_r) - K_r \neq \emptyset$. This implies $K_r \neq \emptyset$. Let $y \in K_r$ such that $f(y) \in X - K_r$. Define $g: R \to R$ such that $g(x) = x$ and note that the linear order $\geq$ on R is such that $(R, \geq)$ is a directed set. Thus, the net $(g, \geq)$ exists.

Let U be an open set such that $y \in U$. For each x in R there exists $p \geq x$ such that $g(p) = p \in U$. Clearly $U - ex$ is open. Since y is a limit point of R, $(U - ex) \cap R \neq \emptyset$. It is possible to pick a $p \in (U - ex) \cap R$, $p \geq x$ and $p = g(p) \in U$. Thus by Kelly (28) page 71, y is a cluster point of the net $(g, \geq)$, and some subnet converges to y. Name it $(g_E, \geq)$ and choose E so that $E \subset R$.

Suppose there exists an $x \in E$ such that $x \in ef(x)$. Since M is cofinal in $R \supset E$, there exists an $m \in M$ such that $x \leq m \leq f(m)$ and the hypothesis of Theorem 3.5 is satisfied. Thus, f has a fixed-point.

Assume $x \in ef(x)$ for every x in E. Since $f(y) \notin K_r$, it may be assumed without loss of generality that $f(x) \notin K_r$ for every x in E. Suppose there exists an x in E such that $f(x) \leq f(f)x))$. In this case the hypothesis of Theorem 3.5 can be satisfied and f has a fixed-point.

The only remaining alternative is $x \leq f(x) \leq f(f(x))$. In this case the hypothesis of Theorem 3.4 is satisfied and f has a fixed-point.

Thus, either $f(K_r) \subset K_r$ for every R or f has a fixed-point. If $f(K_r) \subset K_r$, f has a fixed-point by the hypothesis. Consequently, X has the fpp.

In the proof of Theorem 2.11, the usual sequence of points present in a "dog-chases-rabbit" proof was replaced by a subnet, or what Smithson and Ward call a "generalized sequence." The order used to obtain the net is specifically defined on e-rays. This order and two special properties of M "move" the points along the e-ray to give a chasing effect. The two properties of M which keep the points "moving" are 1) M is cofinal in R, and 2) $m \leq f(m)$ for every $m \in M \cup f(M)$.

The Immediate Technique

When using the "dog-chases-rabbit" technique, it is necessary to define a sequence. In an "immediate" proof, a function rather than a sequence is required. These proofs are called "immediate" because once the function is defined the solution is easily observed. An example of an "immediate" proof is found in K. Kuratowski's proof of Theorem 2.2.

Theorem 2.2. In order that a Peano continuum have the fpp it is necessary that it be unicoherent.

The definition of unicoherent is given in Chapter II as Definition 2.3. The function used in the proof of Theorem 2.2 depends on a quality of Peano continua and an existance theorem also proved by Kuratowski (29) pages 307 and 306. These theorems, Theorems 3.6 and

3.7 are stated here without proof.

Theorem 3.6. Let M be a non-unicoherent Peano space. It is possible to find two Peano continua A and B such that $M = A \cup B$ but $A \cap B$ is not a continuum, (29).

Theorem 3.7. Let P and Q be two nonempty disjoint closed subsets of the continuum M and ab an arc. Then there exists a continuous function mapping M onto ab such that P is mapped onto a and Q is mapped onto b.

A sketch of the proof of Kuratowski's Theorem 2.2 could be:

Theorem 2.2.
Proof:
Suppose M is a Peano continuum which is not unicoherent. By Theorem 3.6 there exist Peano continua A and B such that $M = A \cup B$ and $A \cap B$ is not a continuum. The set $A \cap B = P \cup Q$ where P and Q are closed and disjoint. For p in P and q in Q there exist arcs $ap \subset A$ and $bq \subset B$.
By Theorem 3.7 there exist continuous functions f and g defined on A and B, respectively, such that $f(A) = bq$, $f(P) = q$, $f(Q) = p$, $g(B) = ap$, $g(P) = q$, and $g(Q) = p$. Extend f to $A \cup B$ by letting $f = g$ on B. Assuming f has a fixed-point supplies a contradiction. Consequently, f has no fixed-point and the theorem is proved.

In this particular proof f gives the desired fixed-point because of the relationship between $A \cap B$ and $P \cup Q$. In each theorem using the "immediate" technique, the function defined will be determined by some existing relationships or properties.

A more intricate definition of a function is used in Hamilton's proof of Theorem 2.6.

Theorem 2.6. If D is a bounded simply connected domain in the plane which, together with its boundary, does not separate the plane and whose outer boundary M contains no indecomposable continuum, then every homeomorphism of $\overline{D}$ into itself leaves some point of $\overline{D}$ invariant.

His function is a composition of three functions, two of which are analytic functions having very special properties. These properties are associated with prime ends. The concept of prime end while relatively simple in itself requires several definitions.

Definition 3.3. Let D be a simply connected domain. An arc that lies in D except for its two end-points, or a simple closed curve that lies in D except for one point, is a cross-cut of D.

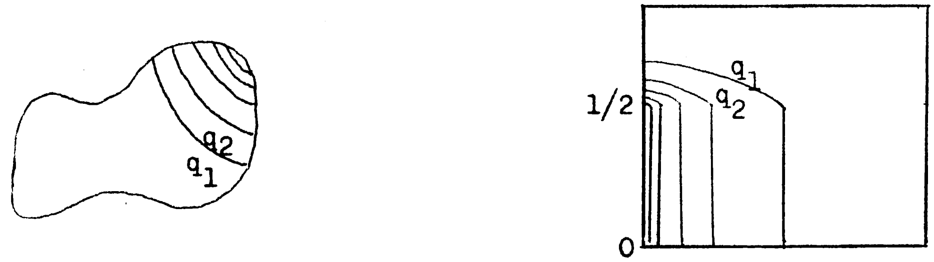
Definition 3.4. A sequence $\{q_i\}$ of cross-cuts of D is called a chain if the following conditions are satisfied:

i) $i \neq j$ implies $q_i \cap q_j = \emptyset$;

ii) q_n separates D into two domains, one of which contains q_{n-1} and the other, d_n, contains q_{n+1}; and

iii) the diameter of q_n tends to zero as $n \to \infty$.

Figure 8 (a) demonstrates d_n such that $\cap \bar{d}_n$ is a single point. In Figure 8 (b) the $\cap \bar{d}_n = \{(0,y)/\ 0 \leq y \leq ½\}$. That $\cap \bar{d}_n$ is either a continuum or a single point is a result in Collingwood and Lohwater (13) page 170.



(a) (b)

Figure 8. The set $\cap \bar{d}_n$

<u>Definition 3.5.</u> Two chains $Q = \{ q_n \}$ and $Q' = \{ q'_n \}$ in D are <u>equivalent</u> if, for all values of n, the domain d_n contains all but a finite number of the cross-cuts q'_n, and the d'_n defined by q'_n contains all but a finite number of the cross-cuts q_n.

<u>Definition 3.6.</u> A <u>prime end</u> of D is an equivalence class of chains in D.

With this definition of prime end it is possible to consider the topology on a simply connected domain D union the prime ends. An open set in this topology shall be a usual open set in D or shall be $G \cup Gp$ where G is a domain defined by a circular cross-cut and does not contain some specified point of D, and Gp is the set of prime ends which have a chain lying (except for the end points) in G. The points in this topology are either usual points of D or prime ends. The prime ends are not points of the plane. The set D with this topology will be denoted by D*. A more complete exposition of the nature of this topology can be found in Collingwood and Lohwater (13), pages 167 - 189.

Since the proof of Theorem 2.6 will deal with prime ends as well as usual points, convergence to a prime end P will be defined.

<u>Definition 3.7.</u> A sequence of points $\{ p_n \}$ or sets $\{ E_n \}$ of D such that every domain d_n contains all but a finite number of them is said to <u>converge to the prime end P</u>.

The remarkable and important theorem of Carathéodory is now intelligible.

<u>Theorem 3.8.</u> Under a one-to-one analytic mapping $w = f(z)$ of the open disk U onto a simply connected domain D, there exists a one-to-one

correspondence between the points $e^{i\Theta}$ in $\{ z / \ |z| = 1 \}$ and the prime ends $P(e^{i\Theta})$ of D*, (13).

Hamilton's function depends upon Theorem 3.8. His proof also requires Theorem 3.9 which was proved by N. E. Rutt (39) page 270.

Theorem 3.9. Let D be a plane bounded, connected and simply connected domain having a boundary M, P a prime end in D*, and $\{ d_n \}$ a sequence of domains associated with P. If M is to be a subset of $\cap \bar{d}_n$ it is necessary that M be either indecomposable or the sum of two indecomposable continua.

Lemma 2 binds together the concepts of inner domain, Definition 2.6, and Definition 2.7. This result will be used in the proof of Theorem 2.6.

Lemma 2. Let D be a bounded simply-connected domain in the plane which together with its boundary does not separate the plane. Then D union its inner boundary points is also a simply connected domain.

Hamilton also uses Theorem 2.5 which he proved himself.

> Theorem 2.5. If M is a compact continuum in the plane which contains no indecomposable continuum, which does not separate the plane, and which contains no domain, then every homeomorphism of M into a subset of itself leaves some point invariant.

In considering the proof of Theorem 2.6 one ought to note how the function of the proof is dependent on the function of Theorem 3.8, and how this gives the desired result because the outer boundary of M contains no indecomposable continuum. A synopsis of the proof follows.

Theorem 2.6.
Proof:

By Lemma 2, the boundary of D may be taken to be its outer boundary.

From Collingwood and Lohwater (13) page 173, there exists an analytic homeomorphism g of the interior I of a given circle J into D such that $g: I \cup J \to D^*$ and g_J is one-to-one and onto the set of prime ends of D*. Letting f represent the homeomorphism of the hypothesis, there exists a homeomorphism $f_1: D^* \to D^*$, determined by f.

Using the properties of g and f, define a function $h: I \cup J \to I \cup J$, $h = g^{-1}f_1g$. This function is a homeomorphism and, by the Brouwer Fixed Point Theorem 1.1, must have a fixed-point.

If h leaves a point of I fixed, then a straight-forward unraveling of $g^{-1}f_1g$ implies f_1 leaves a point of D fixed. By the definition of f_1, the function $f_1 = f$ on D. Therefore, f leaves a point of D fixed.

If h leaves a point of J fixed, the homeomorphism f_1 maps some prime end P into itself. Let $N = \cap \bar{d}_n$, d_n associated with P. Since $f: N \to N$, Theorem 3.9 together with the hypothesis of the present theorem imply that N is a proper subcontinuum of the boundary of D. Since N is a compact continuum which does not separate the plane and which contains no domain, the hypothesis of Theorem 2.5 is satisfied and f leaves fixed some point $N \subset \bar{D}$. This concludes the proof since in either case f leaves some point of D fixed.

In addition to being an example of an "immediate" proof, the proof of Theorem 2.6 illustrates the use of analytic function theory in a topological setting. The function creatively developed by Hamilton in the proof is a composition of the analytic function g, a function f_1 based on the original function f, and g^{-1}. This composition is similar to that used by Hamilton in his proof of the Cartwright-Littlewood Theorem, Theorem 2.9.

Theorem 2.9. If f is a one-to-one continuous and orientation preserving transformation of the Euclidean plane P onto itself which leaves a bounded continuum M invariant and if M does not separate P, then some point of M is left fixed by f.

The function f' constructed in the proof of Theorem 2.9 is developed in Lemma 3. It is an extension of a function which is a composition of three functions: g_1^{-1} which is a homeomorphism, f the function of the hypothesis, and g_2 another homeomorphism. The development of g_1 and g_2 appears simple but, in reality, involves a good many topological tools in the plane.

The theorems used in the proof of Lemma 3 will be listed here to indicate to the reader that the construction of f' resulted from the rather ingenuous pulling together of many properties.

Theorem 3.11. Any compact set which is the common boundary of two domains is a continuum, (51).

Theorem 3.12. If K is a closed subset of a continuum M, and C is a component of S - K, then $K \cap \partial(C) \neq \emptyset$, (49).

Theorem 3.13. Every complementary domain of a connected closed set in the plane has a connected boundary, (37).

Theorem 3.14. Every complementary domain of a continuum in the plane is simply connected, (37).

Definition 3.8. A space or set of points is uniformally locally connected (ulc), if given an $\varepsilon > 0$ there exists a $\delta > 0$ such that all pairs of points, x and y, that satisfy $\rho(x,y) < \delta$ are joined by a connected subset of diameter less than ε .

Theorem 3.15. In a compact space, locally connected means ulc, (37).

Theorem 3.16. Any open set in a locally connected space is itself locally connected, (37).

Theorem 3.17. If a domain in the plane is both simply connected and ulc its boundary is a simple closed curve, or a point, or $\emptyset$, (37).

Theorem 3.18. The components of the boundary of a ulc domain in the plane are all points or simple closed curves, (37).

Theorem 3.19. If D_1 and D_2 are Jordan domains in the plane, a homeomorphism $f: \partial(D_1) \to \partial(D_2)$ can be extended to a homeomorphism of $\overline{D}_1$ onto $\overline{D}_2$, (37).

Theorem 3.20. If the connected point set I_2 does not intersect the boundary of the point set I_1, then either $I_1 \cap I_2 = \emptyset$ or $I_2 \subset I_1$, (35).

Theorem 3.21. If neither of the two intersecting continua H and K separates the space S, each of them has a compact boundary and $H \cup K \neq S$, then every compenent of $H \cap K$ is bounded by a compact continuum, (35).

Theorem 3.22. No locally connected continuum has uncountably many complementary domains, (35).

The function f' in Hamilton's proof of Theorem 2.9,using Theorems 3.11-3.22 as aids, is extended from a topological disk $\overline{\triangle}_1$ to the whole plane on the basis of its definition on $\overline{\triangle}_1$ and basic vector properties. Its extension is in effect a rotation on $P - \overline{\triangle}_1$.

After f' is constructed, the proof of Theorem 2.9 is amazingly short. A powerful tool used is Theorem 3.23, a theorem by Brouwer (9) page 37.

Theorem 3.23. Let p be an arbitrary point of the plane and f a one-to-one continuous orientation preserving transformation of the plane onto itself which leaves no point fixed. Then $\{ f^n(p) / n \in I \}$ has no limit point.

This theorem actually combines two theorems from Brouwer (19), pages 37 - 54. The first of these theorems says that any one-to-one continuous orientation preserving transformation of the plane onto itself which leaves no point fixed is a "translation"; the second theorem, states that for a "translation" the set $\{ f^n(p) / n \in I \}$ has no limit point.

Because it is in Lemma 3 that the function f' in Hamilton's proof of Theorem 2.9 is developed, an abbreviated form of the proof of Lemma 3 will be presented. Following it will be the proof of Theorem 2.9 in its entirety.

Lemma 3. If f is a one-to-one continuous and orientation preserving transformation of the Euclidean plane P onto itself which leaves a bounded continuum M invariant but leaves no point of M fixed and if M does not separate P, then there is a one-to-one continuous and orientation preserving transformation f' of P onto itself which coincides with f on M and leaves no point of P fixed.

Proof:
There exists a simple closed curve C_1 with inner domain D_1 containing M and having the property that if x is in $\overline{D}_1$ then $f(x) \neq x$. Since $\overline{D}_1$ is compact, f is one-to-one, continuous, and onto $f(\overline{D}_1)$, and $f(\overline{D}_1)$ is Hausdorff, f is a homeomorphism from $\overline{D}_1 \to f(\overline{D}_1)$. Let $C_2 = f(C_1)$ and $D_2 = f(D_1)$. The choice of C_1 and the properties of f imply $(C_1 \cup D_1) \not\subset (C_2 \cup D_2)$ and $(C_2 \cup D_2) \not\subset (C_1 \cup D_1)$.

Because neither $D_1 \subset D_2$ nor $D_2 \subset D_1$ and $M \subset \overline{D}_1 \cap \overline{D}_2$, the set $C_1 \cap C_2$ contains at least two points.

Let G be the component of $D_1 \cap D_2$ which contains M. By Theorem 3.21, the $\partial(G)$ is a compact continuum and by Theorem 3.18, the continuum is a simple closed curve. Call this curve J and without loss of generality, assume J to be the unit circle.

Let D_{ri} represent the i-th component of $D_r - \overline{G}$. By Theorem 3.22, the set $\{D_{ri}\}$ is countable for each r, $r = 1, 2, \ldots$ The $\partial(D_{ri})$ is a simple closed curve consisting of two arcs: L_{ri} an arc of J and $a_{ri}b_{ri}$ an arc of C_r. Moreover, $L_{ri} \cap a_{ri}b_{ri} = \{a_{ri}, b_{ri}\}$. For each pair of subscripts r and i, let L_{ri} be a circular arc exterior to G and having endpoints a_{ri} and b_{ri}. Construct L'_{ri} with radius $1 - \delta_{ri}$ where $\delta_{ri} > 0$ is small enough to ensure that no two arcs L'_{ri} meet except at endpoints. This is possible since the arcs L_{ri} of J are disjoint except for end points.

Let $\triangle_{ri}$ be the inner domain of $L_{ri} \cup L'_{ri}$. Define an orientation preserving homeomorphism g_{ri} of $\partial(D_{ri})$ onto $L_{ri} \cup L'_{ri}$ which leaves each point of L_{ri} fixed. Then by Theorem 3.19 g_{ri} can be extended to an orientation preserving homeomorphism of $\overline{D}_{ri}$ onto $\overline{\triangle}_{ri}$.

If there exists an arc $L \subset \partial(\overline{G})$ such that $L \not\subset \partial(D_r)$ for any r and i, then L must be an arc of $C_1 \cap C_2$. In this case consider $\partial(\triangle_{ri}) = L$, $\triangle_{ri} = \emptyset$, $L_{ri} = L$ and g_{ri} the identity map on $L \subset \overline{G}$.

Let $\overline{\triangle}_r = \overline{G} \cup (\cup \overline{\triangle}_{ri})$. Define $g_r\colon \overline{D}_r \to \overline{\triangle}_r$ as follows $g_r(x) = x$ if $x \in \overline{G}$ and $g_r(x) = g_{ri}(x)$ if $x \in \overline{D}_{ri}$. This function g_r is an orientation preserving homeomorphism.

Define $f'\colon \overline{\triangle}_1 \to \overline{\triangle}_2$ as $f' = g_2 f g_1^{-1}$. The mapping f' is an orientation preserving continuous mapping which leaves no points of $\overline{\triangle}_1$ fixed.

Extend f' to all of P. Let z be a point of $P - \overline{\triangle}_1$. Let $\overrightarrow{oz}$ be a vector from the center of J to z. By the construction of $\partial(\overline{\triangle}_1)$, $\overrightarrow{oz} \cap \partial(\overline{\triangle}_1)$ is a unique x and $z = x + \rho u_x$ where u_x is the unit vector in the direction ox and $\rho > 0$. Since $x \in \partial(\overline{\triangle}_1)$, $f'(x) = x'$ is defined and is in $\partial(\overline{\triangle}_2)$. Define $f'(z) = x' + \rho u_{x'}$. This extends f'

to P in such a way that f' is a one-to-one continuous and orientation preserving transformation of P onto itself which coincides with f on M.

Suppose f' has a fixed-point z. Since f' has no fixed-point in $\overline{\triangle}_1$, the point z must belong to P - $\overline{\triangle}_1$ and $z = x + \rho u_x$ for some unique x in $\partial(\triangle_1)$. Assuming $f'(z) = z$ implies u_x and $u_{x'}$ are the same. Therefore, $z = z'$ implies $x = x'$ for x in $\overline{\triangle}_1$. This contradiction implies f' has no fixed-point. Thus f' satisfies the theorem.

Theorem 2.9.
Proof:

Suppose f leaves no point of M fixed. By Lemma 3 there is a one-to-one continuous and orientation preserving function f' of P onto itself such that f' = f on M and f' leaves no point of E fixed. Let p be a point of M. Since f' is one-to-one, has no fixed-point and is a continuous orientation preserving transformation onto itself, Theorem 3.23 implies the sequence $\{f'^n(p)\}$ has no convergent subsequence. This contradicts the fact that M is compact. Therefore, f must leave a point of M fixed.

The Cyclic Element Technique

Thus far the techniques discussed have been based on the definition of a sequence or a function. Neither of these types of proofs depends, in its essence, upon a specific continuum structure. The next type of proof does. It depends on the structure studied in cyclic element theory. As was stated in Chapter II, this theory played an important part in the proofs of Theorem 2.3 and Theorem 2.4.

Theorem 2.3. If the Peano continuum M lies in a plane and does not separate the plane, then every homeomorphism of M into a subset of itself has a fixed-point.

Theorem 2.4. The plane Peano continua which do not separate the plane are characterized by the fixed-point property.

Definitions 3.9 through 3.12 are basic for our discussion of cyclic element theory and its importance in the proofs of Theorems 2.3 and 2.4.

Definition 3.9. A Peano continuum is said to be cyclicly connected provided every two points of M lie together on some simple closed curve which is a subset of M.

Definition 3.10. A nondegenerate cyclicly connected Peano continuum C which is a subset of a Peano continuum M is said to be a maximal cyclic curve of M if and only if C is not a proper subset of any other cyclicly connected Peano continuum which is also a subset of M.

Definition 3.11. A subset E of a Peano continuum M will be called a cyclic element of M provided that E is either i) a maximal cyclic curve of M, ii) a cut point of M, or iii) an end point of M.

Definition 3.12. A property is cyclicly extensible provided that when each cyclic element of the Peano continuum M has this property, M has this property.

A simple closed curve is a Peano continuum which is cyclicly connected and has exactly one maximal cyclic element, namely itself. Example 11 is a Peano continuum consisting of two maximal cyclic curves and infinitely many points which are cyclic elements.

Example 11. Let $M = C_1 \cup C_2 \cup R$ where $C_1 = \{(x,y)/ (x-1)^2 + y^2 = 1\}$ $C_2 = \{(x,y)/ (x+1)^2 + y^2 = 1\}$ and $R = \{(x,0)/ 0 \le x \le ½\}$.

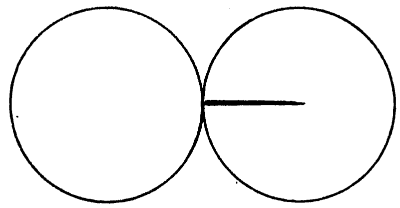


Figure 9. Example 11

In Example 11, C_1 and C_2 are the two maximal cyclic curves. Each point of R is either a cut point or an end point. Example 12 on the other hand, contains infinitely many maximal cyclic curves. Each C_n is a maximal cyclic curve and (1,0) is an end point.

<u>Example 12</u>. Let $M = \bigcup_n C_n \cup \{(1,0)\}$ where $C_n = \{(x,y)/$

$$x - \left(\frac{2^{n+1} - 3}{2^{n+1}}\right)^2 + y^2 = (1/2^{n+1})^2,\ n = 1,2,\ldots\}.$$

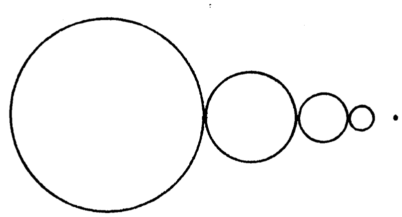


Figure 10. Example 12

A basic and essential theorem regarding the behavior of cyclic elements under homeomorphisms is given by Ayre's (1), page 333, Theorem 3.24.

<u>Theorem 3.24.</u> Let M be a Peano continuum and f a homeomorphism mapping M into M. Then there exists a cyclic element C of M such that $f(C) \subset C$.

The Peano continuum given in the hypothesis of Theorem 2.3 does not separate the plane. A characterization for such Peano continua will be needed. This characterization is stated in Theorem 3.25.

<u>Theorem 3.25.</u> In order that the Peano continuum M should fail to separate the plane it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior, (51).

Since the proof of Theorem 2.3 is short it is presented here in its entirety.

<u>Theorem 2.3.</u>
<u>Proof</u>:
Let f be a homeomorphism mapping M into M. Since M does not separate the plane, Theorem 3.25 implies that every maximal cyclic element of M is a simple closed curve plus its interior. By Definition 3.11, any other cyclic elements of M are points. The Brouwer Fixed Point Theorem 1.1 implies that every cyclic element of M has the fpp. By Theorem 3.24, there exists some cyclic element C of M such that $f(C) \subset C$. Consequently there exists an $x \in C \subset M$ such that $f(x) = x$.

The concluding statements of Theorem 2.3 are equivalent to the statement that if every cyclic element C of the Peano continuum M has the fixed-point property under homeomorphism, then M has the fixed-point property under homeomorphisms. Another way of saying this is to say that the fixed-point property under homeomorphism is cyclicly extensible. Indeed, the fpp itself is cyclicly extensible. This was proven by Borsuk (8) page 205 in 1932. His proof requires a knowledge of retracts. This theory is not in the main stream of the topic of this thesis. Therefore, the proof of the cyclic extensibility of the fpp shall not be included. The result, however, is of such significance that it will be stated as Theorem 3.26.

Theorem 3.26. The fixed-point property is cyclicly extensible.

Using Theorem 3.26 as a lemma, Borsuk was able to prove Theorem 2.4. This theorem generalizes the theorem of Ayres' just discussed. Like the proof of Ayres' Theorem 2.3, Borsuk's proof is short and directly dependent on the cyclic structure of Peano continua. It is presented here in its complete form.

Theorem 2.4.
Proof:
Let M be a Peano continuum which separates the plane. Then M is not unicoherent and Kuratowski's Theorem 2.2 implies M does not have the fpp.

Let M be a Peano continuum which does not separate the plane. By Theorem 3.25 every maximal cyclic element is a simple closed curve plus its interior. Such sets have the fpp by Theorem 1.1. Consequently, every cyclic element of M has the fpp and Theorem 3.26 implies M has the fpp.

The Change of Topology Technique

The fourth technique to be exposed in this chapter is called "change of topology." It is a method seldom encountered in fixed-point theorems. In the proof of Hamilton's Theorem 2.6, a change of topology was induced by the prime ends. The proof, however, depended upon the construction of a specific function exploiting the properties of prime ends. In Young's "change of topology" proof for Theorem 2.7 the new topology leads directly to the desired result.

Theorem 2.7. Let M be an arcwise connected Hausdorff space which is such that every monotone increasing

sequence of arcs is contained in an arc. Then M has
the fixed-point property.

A "change of topology" on a space S is effected by a collection G of subsets of S. The following definition determines G-open and G-closed sets.

Definition 3.13. Let G be a collection of subsets of some given topological space S. A point p of S is a G-limit point of a subset M of S provided that every open neighborhood of p contains an element V of G such that $p \in V$ and $V \cap (M - p) \neq \emptyset$. A set is G-closed if it contains all its G-limit points. A set is G-open if its complement is G-closed.

In the proof of Theorem 2.7 the set G is the collection of all arcs of the space M. It can be verified that G along with Definition 3.13 determines what will be called the arc-topology of M. The motivation for such a change in topology lies in the fact that in the arc-topology M is locally connected. With the aid of Theorem 3.27, this fact will be proven in the proof of Theorem 2.7.

Theorem 3.27. If every element of G is G-connected and S is a T_0 space then S is G-locally connected, (53).

The change of topology which will be encountered in the proof of Theorem 2.7 makes M a generalized dendrite as defined in Definition 3.14. For the definition of chain, see Definition 2.9.

Definition 3.14. By a generalized dendrite is meant a locally connected Hausdorff space S such that if a,b are two points of S, and C and C' are two chains of connected open sets from a to b and C has more than two links, then some link of C that does not contain a or b intersects some link of C'.

In the paper entitled "The Introduction of Local Connectivity by Change of Topology" in which Theorem 2.7 appears, Young (53) pages 481 and 485, proves several theorems regarding G-topologies. Two of these results are needed for the proof of Theorem 2.7.

<u>Theorem 3.28.</u> If S is a Hausdorff space in the original topology, it is also a Hausdorff space in the G-topology.

<u>Theorem 3.29.</u> If a transformation f defined on a T_o space S is continuous in the original topology, it is G-continuous for any collection G.

In the same paper, Young (53) page 491, proves Theorem 3.29 regarding generalized dendrites. This theorem which is needed for the proof of Theorem 2.7 is a generalization of an earlier theorem by Wallace (51) for certain point-to-set mappings.

<u>Theorem 3.30.</u> Let the separable space S be a generalized dendrite. If the union of any increasing sequence of arcs of S is contained in an arc, then S has the fixed-point property.

A shortened proof of Theorem 2.7 follows.

<u>Theorem 2.7.</u>
<u>Proof:</u>
The hypothesis implies M contains no simple closed curve in either its original topology or in its arc-topology. By Theorem 3.28, M is Hausdorff in the arc-topology. From Definition 3.14 and Theorem 3.27 it follows that M is arc-locally connected.

Let C and C' be two chains of connected open sets joining the two points a and b in the arc-topology, C having more than two links. Because M is arcwise connected some link of C not containing a or b must intersect some link of C'. Thus M satisfies Definition 3.15 and is a generalized dendrite in the arc-topology.

Since a Hausdorff space is T_0, the mapping f is continuous in both the original and the arc-topology by Theorem 3.29. Thus, f is a continuous mapping of M into M with the arc-topology in which M is a generalized dendrite. By Theorem 3.30, f has a fixed point.

The Sequence of Arcs Technique

Another method used in proving fixed-point theorems is called the "sequence of arcs" approach. Mohler's Theorem 2.18 is the only theorem included in Chapter II which makes use of this technique.

Theorem 2.18. If H is a hereditarily unicoherent continuum and $\tau(H) \neq \infty$, then H has the fixed-point property.

The goal in this type of proof is to show that a specified set, not necessarily the entire continuum, meets the requirements of Theorem 2.7 which was just discussed.

Various properties of arcs are needed in a "sequence of arcs" proof. Those properties used in the proof of Theorem 2.18 are stated here as Theorems 3.31-3.37. The sets N(H) and J(H) appearing in Theorems 3.31 and 3.32 are defined in Definitions 2.21 and 2.22, respectively.

Theorem 3.31. If H_1 and H_2 are hereditarily unicoherent continua and $H_1 \subset H_2$, then $N(H_1) \subset N(H_2)$, (34).

Theorem 3.32. If $\beta < \alpha$, then $J^{\alpha}(H) \subset J^{\beta}(H)$, (34).

Theorem 3.33. If H is a hereditarily unicoherent continuum such that $\tau(H) \neq \infty$ and if f: $H \Rightarrow H$ is a fixed-point free map, then H contains a subcontinuum H' such that $\tau(H') \neq \infty$ and such that $f_{H'}$ maps H' onto

H', (34).

<u>Theorem 3.34.</u> Let A be an arcwise connected subset of the hereditarily unicoherent continuum X and let $\{a_n b_n\}$ be a sequence of monotone increasing arcs of A. If $p \in a_n b_n$, $p \neq b_n$, $p \neq a_n$ for some n, then $\overline{\cup pb_n}$ or $\overline{\cup a_n p}$ is an arc, (34).

<u>Theorem 3.35.</u> Let A and $\{a_n b_n\}$ be as in Theorem 3.34. If $p \in \cup a_n b_n$ and $y \in pb_n - \{p, b_n\}$ for some p such that $p \in a_n b_n$, then $py \cup (\overline{\cup\ yb_n}) = \overline{\cup pb_n}$ and $py \cap (\overline{\cup yb_n}) = y$. A similar statement holds for $\cup a_n p$, (34).

<u>Theorem 3.36.</u> Let A and $\{a_n b_n\}$ be defined as in Theorem 3.34 and let $p \in \cup a_n b_n$. If $\overline{\cup pb_n}$ is locally connected at some $z \in \overline{\cup pb_n} - (\cup pb_n)$, then $\overline{\cup pb_n}$ is an arc. A similar statement holds for $\overline{\cup a_n p}$, (34).

<u>Theorem 3.37.</u> Let A and $a_n b_n$ be as in Theorem 3.34. If $x \in \cup a_n b_n$ and $z \in \overline{\cup xb_n} - \cup xb_n$ then $\overline{\cup xb_n}$ is irreducible between x and z. A similar statement holds for $\overline{\cup a_n x} - a_n x$, (34).

As was mentioned in Chapter II, Mohler proved a generalized form of Young's Theorem 2.7 to effect the proof of Theorem 2.18. The proof of Theorem 2.18 as given here, however, is restricted to the plane and Young's theorem is sufficient.

<u>Theorem 2.18.</u>
<u>Proof:</u>
Suppose H does not have the fixed-point property. Then by Theorem 3.33, H contains a subcontinuum H' with $\tau(H') \neq \infty$ and such that $f_{H'}$ maps H' onto H'. Consequently, it is sufficient to consider H a fixed hereditarily unicoherent continuum for which $\tau(H) = h$ and $f: H \to H$ is an onto map.

Let x denote a particular point in $J^h(H)$ and let A denote the set of all points in H which can be joined to x by an arc in H. Then A is arcwise connected. Using Young's Theorem 2.7, it will suffice to show that every monotone increasing sequence of arcs in A is contained in an arc in A.

Let $\{a_n b_n\}$ be a sequence of monotone increasing arcs in A and let $p \in \bigcup a_n b_n$, $p \neq a_n$, $p \neq b_n$ for some n, then by Theorem 3.34 it can be assumed that $\overline{\bigcup pb_n}$ is an arc. That is, there exists a $y \in \overline{\bigcup a_n b_n}$ such that $py = \overline{\bigcup pb_n}$. By Theorem 3.35, $(\overline{\bigcup a_n p}) \cup (\overline{\bigcup pb_n}) = (\overline{\bigcup a_n p}) \cup (py) = \overline{\bigcup a_n y} \subset A$. If it can be shown that $\overline{\bigcup a_n y}$ is an arc, the proof will be complete.

Suppose $\bigcup a_n y$ is not an arc. Then by Theorem 3.36, $\overline{\bigcup a_n y}$ fails to be locally connected at every point of the set $L(y) = \overline{\bigcup a_n y} - \bigcup a_n y$. Since $\tau(H) = h \neq \infty$ there exists a $\beta = \sup\{\gamma / L(y) \subset J^{\gamma}(H)\}$. Thus, $L(y) \subset J^{\beta}(H)$ and $L(y) \not\subset J^{\beta+1}(H)$. The definition of J^{β} implies $J^{\beta}(H)$ is locally connected at some $z \in L(y)$. It follows that $J^{\beta}(H) \cap (\bigcup a_n y) = \emptyset$.

Let xy be the unique arc joining y to the point x by which A was determined. Theorem 3.32 implies $J^h(H) \subset J^{\beta}(H)$. Therefore, $x \in J^{\beta}(H)$. Thus $\{x, y\} \subset J^{\beta}(H) \cup (\bigcup a_n y)$ which is connected. It follows that $xy = I(\{x, y\}) \subset J^{\beta}(H) \cup (\bigcup a_n y)$.

The set $xy \cap (\overline{\bigcup a_n y}) = \overline{\quad a_n y}$ or, equivalently $\overline{\bigcup a_n y}$ is an arc. This completes the proof since by Theorem 2.7, $A \subset H$ has a fixed-point.

Summary and Conclusions

Five fundamental types of proofs have been exposed in Chapter III; the "dog-chases-rabbit", the "immediate", the cyclic element, the "change of topology", and the "sequence of arcs." These five techniques emerged from the analyzation of Theorems 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, and 2.18.

In the study of these fixed-point theorems it became clear that the approach used in each has its unique requirement. The "dog-chases-rabbit" technique, or a variation of it, requires a sequence of points $\{x_n\}$ such that for every $\varepsilon > 0$ there exists an n such that $\rho(x_n, f(x_n)) < \varepsilon$. The "immediate" type proof involves a function from which the result is easily obtained. The cyclic element approach is valid only for Peano continua. The "change of topology" is determined by a collection of subsets of the given continuum. In the "sequence of arcs" approach, every sequence of monotone increasing arcs of the continuum or a subcontinuum must be contained in an arc.

In the past, the "dog-chases-rabbit" and the "immediate" approaches have allowed for the most variation and have provided the widest spectrum of proofs. They are still viable approaches for the future.

Because it is now known that non-separating plane Peano continua have the fpp, the usefulness of the cyclic element theory approach has been exhausted. However, the strategy of using a particular technique for particular types of continua is still a realistic method of proof.

The last two techniques are related in that the last one uses the theorem, Theorem 2.7, proved by the first. Neither of these approaches has been widely used or explored.

If the study of fixed-point techniques were to end with Chapter III, it would be incomplete. H. Bell's Theorem 2.14 uses a technique which, because of its complexity, is classified as an approach in its own right. This approach, along with theorems dependent on Bell's theorem, is exposed in Chapter IV.

CHAPTER IV

AN EXPOSITION OF BELL'S PAPER AND DEPENDENT RESULTS

Introduction

The first part of this chapter contains an exposition of Bell's proof of Theorem 2.13 restricted to Theorem 2.14. The last part of the chapter contains an exposition of theorems directly dependent on Theorem 2.14.

> Theorem 2.14. Let M be a bounded plane continuum which does not separate the plane, and let g by a locally bounded function defined on the plane such that $g(x) \in M$ if $x \notin \mathrm{Int}(M)$. If g is continuous at each point of M, then either g_c has a fixed-point in M or there is an indecomposable continuum $Q \subset M$ such that $Q = g(Q)$.

In the exposition of the proof of Theorem 2.14 an effort is made to improve the readability. Theorems 4.1-4.13 are stated as lemmas in Bell's (3) paper "On Fixed Point Properties of Plane Continua" as it appeared in the Transactions of the American Mathematical Society, 1967. Bell did not include the proofs of Theorems 4.3, 4.4, and 4.5. These proofs, along with more complete explanations within Bell's proofs, are part of the exposition presented here.

In order to facilitate the flow of proofs, several useful properties are stated and proved in the form of lemmas. Figures are included to illustrate unfamiliar definitions. In addition, between theorems paragraphs of motivational and interlinking prose are presented.

The order in the exposition of Bell's work is basically the order he followed. The steps to be taken are:

1. Develop the concept of $T(Q)$.

2. Define and illustrate S_n, F_n, K_n, A_n, and D_n. These are constructed so that D_n is a simple closed curve, $T(Q) = \cap T(D_n)$ and $T(D_{n+1}) \subset T(D_n)$.

3. Prove that if U is an open proper subset of the plane that contains a plane continuum M which does not separate the plane, then there is a simple closed curve D such that $T(M) = M \subset \mathrm{Int}(T(D))$ and $T(D) \subset U$.

4. Develop the concept of locally bounded fuction and $f_c(x)$.

5. Prove that for Q a plane continuum, g a locally bounded function defined on the entire plane such that g is continuous at each point of Q and $g(x) \in T(Q)$, and D a simple closed curve for which $Q \subset T(D)$, g_c has a fixed-point in $T(D)$ or $D \cap \{x/x \notin T(Q)$ and $x \in T[Q \cup g_c(x)]\} \neq \emptyset$.

6. Prove a sufficient condition for a plane bounded continuum to be indecomposable.

7. Apply the six preceding steps to the case where M is a bounded plane continuum which does not separate the plane, g is a locally bounded function defined on the plane such that $g(x) \in M$, and Q is either a proper subcontinuum of $\partial(M)$ for which $g(Q) \subset Q$ or $Q = \partial[T(M)]$ if no such proper subcontinuum exists.

8. Let f be an extension of the restriction of g to $T(Q)$ such that f is continuous at each point of Q and show that there exists an n' such that for $n \geq n'$, A_n contains no fixed-points of f_c.

9. Define K", Y, J_n, H_n and accessible point of Y.

10. Use the previous definitions and theorems to prove a) $y \in A(Y)$, the set of accessible points of Y, if and only if there exists an

$n \geq n'$ for which $y \in H_n - T(Q)$, b) $A(Y) \subset K''$, c) $A(Y)$ contains no simple closed curve, d) $A(Y)$ is open relative to $\overline{A(Y)}$, and e) $\overline{A(Y)} - A(Y) \subset Q$.

11. Define L as $L' \cup L(y')$ where y' is a specified accessible point of Y, $L(y')$ is one of the components of $A(Y) - y'$ having y' as an end-point, and L' is a ray having $y' = L' \cap Y$ as an end point.

12. Use previous steps to prove that $\overline{L} - L = Q = f(\overline{L} - L)$ and that Q is indecomposable.

Basic Understandings

The proofs in this chapter assume considerable knowledge of the topology of the plane. To include a list of theorems employed would make the exposition unwieldy. However, Moore (35) and Newman (37) are references which the reader will find helpful.

The set M will be a bounded plane continuum which does not separate the plane. The set Q will be a bounded plane continuum. In the proof of Theorem 2.14, Q will be chosen as follows: if the boundary of M contains a proper subcontinuum W for which $g(W) \subset W$, then Q will be the minimal such; if no such W exists, Q will be the boundary of $T(M)$.

When M is a bounded plane continuum which does not separate the plane, $T(M) = M$. The set $T(Q)$ where Q is a bounded plane continuum plays an important part in the proof of Theorem 2.14. For that reason, it will be helpful to have available the following property of $T(Q)$.

Lemma 4.1. Let C be an arbitrary bounded and closed set. Then $\partial[T(C)] \subset C$ and $T(C)$ is a closed set.

<u>Proof</u>: If $C = T(C)$, the proof is complete. If $C \neq T(C)$, then $P - C = A \cup B$ sep. Since C is bounded, it may be assumed without loss of generality that B is bounded and A is unbounded. By Definition 2.15, $T(C) = C \cup B$.

Let x be a boundary point of $C \cup B$ such that $x \notin C$. Then $x \notin A$. For if $x \in A$, the definition of boundary point implies x is a limit point $C \cup B$. By Moore (35) page 5, x is a limit point of C or x is a limit point of B. This is impossible since C is closed with $C \cap A = \emptyset$ and $A \cap \overline{B} = \emptyset$. Thus, $x \in B$.

By the definition of boundary point, every open set U containing x contains a point y not in B. Since A and B are separated sets, there exists an open set U containing x such that $U \cap A = \emptyset$. This contradicts x being a boundary point of $C \cup B$. Thus, for $x \in \partial[T(C)]$, $x \in C$. By a similar argument $\overline{T(C)} = T(C)$.//

Bell's proof of Theorem 2.14 requires the use of a particular descending chain of two-cells. This idea is based on the fact that every nonseparating plane continuum is the intersection of a descending chain of topological two-cells. The appropriate definitions for developing Bell's two-cells will now be introduced.

<u>Definition 4.1.</u> For each natural number n let $\underline{S_n}$ be the collection of square two-cells consisting of points of the form (a,b) where $k/2^n \leq a \leq (k+1)/2^n, j/2^n \leq b \leq (j+1)/2^n$ with k and j intergers.

<u>Definition 4.2.</u> Let S_n be as in Definition 4.1. For each natural number n, let $\underline{F_n}$ be the collection of the boundaries of the two-cells S_n.

<u>Definition 4.3.</u> Let F_n be as in Definition 4.2. For each natural number n, let $\underline{K_n}$ be the collection of open arcs that are components of some $F - T(Q)$ where $F \in F_n$, $F \not\subset T(Q)$ and $F \cap T(Q) \neq \emptyset$.

The set $K_n \neq \emptyset$ for some n. This can be argued in the following manner. Let x by the "highest" point of T(Q). The construction of the S_n's implies there exists an n such that for any particular $S \in S_n$, $T(Q) \not\subset S$. If $x \in F$ for some $S \in S_n$, then the S with x in its bottom or side boundaries has $F - T(Q)$ in K_n. If x is in the interior of some $S \in S_n$, then by Theorem 3.20, $T(Q) \cap F \neq \emptyset$. Arguing as before, $K_n \neq \emptyset$.

<u>Definition 4.4.</u> Let S_n be as in Definition 4.1. Let $A_n = T[\cup\{S \in S_n / S \cap Q \neq \emptyset\}]$ and let D_n be the boundary of A_n.

In Figure 11 the preceding definitions are illustrated for n = 1 with M an arc. This is, of course, an easily pictured and extremely simple case.

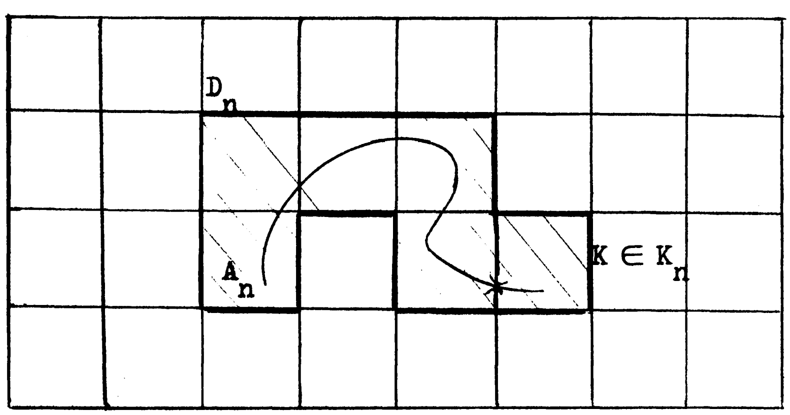


Figure 11. Illustration of Definitions 4.1-4.4

Based on Definitions 4.1-4.4, Bell proved three theorems regarding the K_n's, the D_n's and the relationships between $T(D_n)$ and $T(Q)$. These theorems are stated as Theorems 4.1, 4.2, and 4.3. The understanding of the concepts in these theorems is basic to the understanding of Bell's proof.

<u>Theorem 4.1.</u> Let D_n be defined as in Definition 4.4. Each D_n is a simple closed curve.

<u>Proof</u>: Let n be a fixed natural number. The definition of A_n implies Q is contained in the interior of A_n. From the definition of D_n it is clear that D_n contains a simple closed curve. Call that curve D. Since Q is contained in the interior of A_n, the set $D \cap Q = \emptyset$. By Theorem 3.20, either $Q \subset T(D)$ or $Q \cap T(D) = \emptyset$.

Suppose I is a side of some $S \in S_n$ contained in D. The set $I \cap Q = \emptyset$ since $I \subset D$. The definition of A_n implies I is a side of some S" such that $S" \subset T(D)$. The construction of the S_n's, however, implies I is also the side of an S' such that $S' \not\subset A_n$. Since $S' \not\subset A_n$, $S' \cap Q = \emptyset$. By definition of D_n and A_n, the set $S" \subset A_n$. Suppose $S" \cap Q = \emptyset$.

If $S" \cap Q = \emptyset$, then $Q \subset A_n - (S' \cup S") = H$. It will be shown that H does not separate the plane. Suppose the opposite, namely, $P - H = A \cup B$ sep. Since $S' \cup S"$ is connected, it can be assumed that $(S' \cup S") \subset B$. Thus $P - A_n = A \cup [B - (S' \cup S")]$ sep. or A_n separates the plane. This contradicts the definition of A_n. Therefore, H does not separate the plane. But $\{ S \in S_n / S \cap Q \neq \emptyset \} \subset H$ implies $H \subset A_n$. This contradicts A_n being the smallest bounded subset containing Q and not separating the plane. Consequently, $S" \cap Q \neq \emptyset$, and by Theorem

3.20, $Q \subset T(D)$.

By definition and the assumption that $D \subset D_n$, $A_n = T[\cup\{S/S \in S_n,$ $S \cap Q \neq \emptyset\}] \subset T(D) \subset T(A_n) = A_n$. Thus $T(D) = A_n$ and D_n is the boundary of $T(D)$ which is the simple closed curve D.//

To facilitate the proof of Theorem 4.2, a ray will be defined.

Definition 4.5. A set L will be called a ray if there is a homeomorphism, h, of the set of nonnegative real numbers onto L such that $\lim_{x \to \infty} |h(x)| = \infty$. The point h(0) will be called the endpoint of the ray.

Theorem 4.2. Let D_n be defined as in Definition 4.4. The continuum $T(Q) = \cap T(D_n)$ and $T(D_n) \supset T(D_{n+1})$ for each natural number n.

Proof: The definition of A_n implies $T(Q) \subset \cap T(D_n)$. Suppose $x \notin T(Q)$. Since $T(Q)$ is bounded and does not separate the plane there exists a ray L with endpoint x, such that $L \cap T(Q) = \emptyset$. Since L and Q are each closed there exists an N such that $1/2^N < \rho(L,Q)$. Consequently, for every $S \in S_N$, $S \cap Q = \emptyset$ or $S \cap L = \emptyset$. But this implies that $A_n \cap L = \emptyset$. The definition of A_n implies $D_n \subset F_n$. In fact, $D_n \subset \{F/ F$ is the boundary of $S \in S_n$ where $S \cap Q \neq \emptyset\}$. Thus $x \notin A_n \cup D_n$, or since D_n is a simple closed curve, $x \notin T(D_n)$. So, if $x \notin T(Q)$, $x \notin \cap T(D_n)$.

The definition of D_n implies that $T(D_n) \supset T(D_{n+1})$ for every natural number n.//

Theorem 4.3. Let $K' \subset \cup K_n$ where K_n is defined as in Definition 4.3. Let j be a fixed natural number, then

i) $T(\cup\{K/ K \in K'\} \cup Q) = \bigcap_{n=j}^{\infty} T(\cup\{K/ K \in K'\} \cup D_n)$,

ii) $\cup\{K/ K \in K'\} \cup Q$ is closed, and

iii) the boundary of $T(\cup\{K/\ K \in K'\} \cup D_j)$ is a simple closed curve.

Proof: (i) Since $(\cup\{K/\ K \in K' \cup Q)\} \subset (\cup\{K/\ K \in K'\} \cup D_n)$ for every n, the left hand side is a subset of the right hand side.

Suppose $x \in T(\cup\{K/\ K \in K'\} \cup D_n)$ for every $n > j$. Properties of the plane along with the definition of K_n and Theorem 4.1 imply $\cup\{\bar{K}/\ K \in K',\ K \not\subset T(D_n)\} \cup D_n = H$ is a continuum for every $n > j$. By Definition 2.15 and the definition of A_n, $T(H) = H \cup A_n \cup T[\cup\{K/\ K \in K', K \not\subset T(D_n)\}]$. If $x \in T[\cup\{K/\ K \in K',\ K \not\subset T(D_n)\}]$, $x \in T(\cup\{K/\ K \in K'\} \cup Q)$. If $x \in H$ then $x \in \{\bar{K}/\ K \in K',\ K \not\subset T(D_n)\}$ or $x \in D_n$ for every $n > j$. In either case $x \in T(\cup\{K/\ K \in K'\} \cup Q)$. If $x \notin H$, then $x \in A_n$ for every $n > j$. By the definition of A_n, $x \in T(D_n)$ for every $n > j$ and by Theorem 4.2, $x \in Q$. Thus if x is in the right hand side of (i), x is in the left hand side of the equality.

(ii) At level n, A_n contains all but a finite number of K's from K'. From Lemma 4.1 and the definition of D_n, the set $T(D_n)$ is a closed set containing A_n. Moreover, $T(Q) \subset T(D_n)$ implies $T(D_n)$ contains the end points of K for every $K \in K'$. Therefore, $T(Dn) \cup \{K/\ K \in K'\} - A_n$ is closed and equals $T(D_n) \cup \{K/\ K \in K'\}$. The compactness of $D_1 \cup \{K/\ K \in K'\}$ implies $\cap[T(D_n) \cup \{K/\ K \in K'\}] = \cap[T(D_n)] \cup [\cap\{K/\ K \in K'\}] = M \cup \{K/\ K \in K'\}$ is a closed nonempty set.

(iii) By the definition of K', the set A_j contains all but a finite number of the K's in K'. The set $D_j \cap K$ is finite since D_j is a union of arcs of length $1/j$, the maximum length of K is $1/n$, and there exists a number b such that $b(1/j) = 1/n$. Without loss of generality let K be a subarc of the original K formed by $K - D_j$. It will be shown first that $T(K \cup D_j)$ is a simple closed curve for a single $K \in K'$. By definition, K is an arc with endpoints p and q in $T(D_j)$. There exists a point $x \in K$ such that $x \notin T(D_j)$. By Theorem 3.20, the set $px \cap D_j \neq \emptyset$ and the set $qx \cap D_j \neq \emptyset$. Let h and k belong to $px \cap D_j$ and $qx \cap D_j$, respectively, $h \neq k$. Let h'k' be the arc of D_j such that $Q \subset (h'k' \cup hk) = H$, hk the arc from h to k in K. By elementary arc properties it follows that for every $a,b \in H$, H can be expressed as the union of two independent arcs from a to b. By Hall and Spencer (19) page 171, H is a simple closed curve. The fact that H is the boundary of $T(K \cup D_j)$ follows from the definition of D_j, K and Definition 2.15.

Since the only properties of D_j necessary for the proof were a) D_j is a simple closed curve and b) $T(D_j)$ contains the end points of K, the set $K \cup D_j$ can now be substituted for D_j. In this way using finite induction, it can be shown that the boundary of $(\cup\{K/\ K \in K'\} \cup D_j$ is a simple closed curve.//

The importance of Theorems 4.1-4.3 will become clearer as the chapter progresses. At this point some importance is given to Theorem 4.1 by its use in proving Theorem 4.4.

<u>Theorem 4.4.</u> If U is an open proper subset of the plane that contains a plane continuum M which does not separate the plane, then there is a simple closed curve D such that $T(M) = M$ is contained in the interior of $T(D)$ and $T(D) \subset U$.

<u>Proof:</u> Let $x \in M \subset U$. Since U is open there exists an open set U_x such that $x \in U_x$ and $U_x \subset U$. The regularity of the plane implies there exists an n such that for some subset S* of S_n, $S^* \subset U_x$ and $x \in \mathrm{Int}(S^*)$. The set $S" = \{\ \mathrm{Int}(S^*)/\ x \in \mathrm{Int}(S^*)$ for $x \in M\ \}$ covers M. The compactness of M implies there exists a finite subcover $S' \subset S"$

such that $M \subset S'$. Let N be the maximum of $\{ n / S \in S_n, \mathrm{Int}(S) \subset S'' \}$. For some $n > N$, there exists S_n such that for $x \in M$, $S \in S_n$, $x \in S$, the set $S \subset [\cup \{ S^* / S^* \in S'' \}]$. Let H be the union of such S's, then $H \subset U$. By Definition 2.15, $T(H) = H \cup A$ where A is a set bounded by H. Thus, $T(H) \subset U$. But by definition $T(H) = A_n$. Therefore, $\partial[T(H)] = D_n \subset U$ with D_n a simple closed curve by Theorem 4.1.//

Besides defining the topological two-cells, Bell defined and used the concept of a locally bounded function, Definition 2.16. As was stated in Chapter II, Definition 2.18, a continuous function f defined on Q can be extended to a locally bounded function g on the entire plane. For the rest of this chapter a locally bounded function defined on the entire plane will be an f so extended. With this extension the g of Theorem 2.14 has $g(x) \in M$ for every x. Therefore, the phrase "if $x \in \mathrm{Int}(M)$" can be omitted from Theorem 2.14.

The set f_c was defined in Definition 2.17. Figure 12 illustrates $f_c(x)$ on the continuum Q where Q_2 is a homeomorphic image of Q_1, $f(x_1) = y_1$ and $f(x_2) = y_2$.

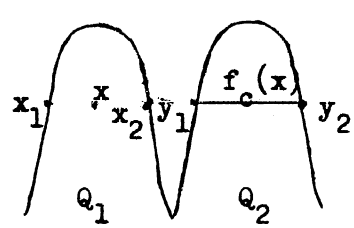


Figure 12. The Set $f_c(x)$

A basic, highly useful property of $f_c(a)$ is stated in Lemma 4.2.

Lemma 4.2. Let $f_c(a)$ be defined as in Definition 2.17. If there exists a compact set G such that $\overline{C(f(U \cap A))} \subset G$ for some U, then $f_c(a)$ is a nonempty closed set.

Proof: Let U be an open set such that $\overline{C(f(U \cap A))}$ is contained in some compact set G. Assume $\delta(U \cap A) = \varepsilon$. Let $\{U_n\}$ be a sequence of open sets such that $a \in U_n$, $U_n \subset U$ and $\delta(U_n) = \varepsilon/n$. The $\overline{C(f(U_n \cap A))} = Z_n$ is closed in G and $Z_n \supset Z_{n+1}$. Thus $\{Z_n\}$ is a sequence of closed sets having the finite intersection property. By Hall and Spencer (19) page 68, the set $\cap Z_n \neq \emptyset$. Since the intersection of any collection of closed sets is a closed set, $f_c(a)$ is a nonempty closed set.//

For f an lb extension, $f(x) \in T(Q)$ for every x in the plane. Since $T(Q)$ is bounded, $\overline{C(f(U \cap A))}$ is bounded for every U. Thus, $f_c(x) \neq \emptyset$ for every x in the plane. It is possible, therefore, to consider f_c as a multivalued function defined on the plane. The importance of f_c will become evident as Bell's proof of Theorem 2.14 is pursued. Its importance results from Lemma 4.2 and from its relationship to continuous and uppersemicontinuous multivalued functions. These relationships are stated and proved in Theorems 4.5 and 4.6.

Theorem 4.5. Let f be a locally bounded function defined on a set A. Then f is continuous at a point x in A if and only if $f_c(x) = f(x)$.

Proof: Let f be continuous at x and suppose $f_c(x) \neq f(x)$. Then $\delta[f_c(x)] = \varepsilon > 0$. Let U be an open set about f(x) with diameter $\varepsilon/10$. Since f is continuous there exists an open set V such that

$f(V \cap A) \subset U$. According to Hocking and Young (23) page 207, the diameter of the convex hull of a set is equal to the diameter of the set. Therefore, $\delta\{\overline{C[f(V \cap A)]}\} = \delta(f(V \cap A)) \leq \delta(U) = \varepsilon/10$. But $\delta(f_c(x)) = \varepsilon$ implies $\delta(C[f(V \cap A)]) \geq \varepsilon$. This contradiction completes this half of the proof.

Let $f_c(x) = f(x)$ and let V be an open set containing $f(x)$. For each n, the open set U_n contains x and $\delta(U_n \cap A) = 1/n$, $U_{n+1} \subset U_n$. Consequently, $f(U_{n+1} \cap A) \subset f(U_n \cap A)$ and $\overline{C[f(U_{n+1} \cap A)]} \subset \overline{C[f(U_n \cap A)]}$.

For some N', $f(U_{N'} \cap A) \subset V$. If this were not true, then for every n, the set $[f(U_n \cap A)] - V \neq \emptyset$. Since $\overline{C[f(U_n \cap A)]} \supset f(U_n \cap A)$, the set $\overline{C[f(U_n \cap A)]} - V \neq \emptyset$ for every n. By definition the set $\overline{C[f(U_n \cap A)]}$ is closed and $\overline{C[f(U \cap A)]} - V \subset \overline{C[f(U_n \cap A]} - V$, each of which is closed for every n. Because f is locally bounded, the set $C[f(U_n \cap A)]$ is bounded for every n. In particular, $\overline{C[f(U_1 \cap A)]}$ is closed and bounded, or compact. This assures that $\cap\{\overline{C[f(U_n \cap A)]} - V\} = K \neq \emptyset$. But $K \subset f_c(x) - f(x) = \emptyset$. This contradiction completes the proof.//

If one begins with a continuous function f defined on a bounded continuum Q and f is lb extended to a locally bounded function g defined on all the plane, Theorem 4.5 states that $g_c(x) = f(x)$ on Q. Thus, one can consider whether or not $g_c(x)$ has a fixed-point in Q to determine whether or not f has a fixed-point in Q. This is part of Bell's approach.

The next theorem states that a locally bounded function is an upper semicontinuous multivalued function.

<u>Theorem 4.6.</u> Let f be a locally bounded function defined on a set A,

then f_c is an upper semicontinuous multivalued function; that is, if $x_n \to x$ and $y_n \to y$ where $y_n \in f_c(x_n)$, then $y \in f_c(x)$.

Proof: For every n let U_n be a spherical neighborhood containing x such that $\delta(U_n) = 1/n$. Let U be an arbitrary neighborhood containing x. For some n', $U_{n'} \subset U$. Since $x_n \to x$, there exists an N such that for every $n > N$, $x_n \in U_{n'} \subset U$. From the hypothesis $y_n \in f_c(x_n)$. Therefore, $y_n \in \overline{C\,[f(U \cap A)]}$ for every $n > N$. Since $y_n \to y$ and $\overline{C\,[f(U \cap A)]}$ is closed, $y \in \overline{C\,[f(U \cap A)]}$. The neighborhood U being arbitrary implies $y \in \overline{C\,[f(U \cap A)]}$ for every U that is open and $x \in U$. Thus, $y \in f_c(x)$.//

Preliminary Theorems

Because f_c is an upper semicontinuous multivalued function, Bell was able to generalize a result of Eilenberg and Montgomery (15). His generalization is stated as Theorem 4.7. Its proof as given by Bell does not depend on the work of Eilenberg and Montgomery.

Theorem 4.7. Let D be a simple closed curve and let f be a bounded function defined on T(D) for which $f(D) \subset T(D)$. If for each point $d \in D$ there is an open set U containing d such that for any two points e and h contained in $U \cap D$ there is an arc A, contained in T(D), joining f(e) to f(h), for which $T(A \cup [f(e)f(h)]) \cap U = \emptyset$, then f_c has a fixed-point.

Figure 13 is an attempt to illustrate the situation described in Theorem 4.7. Bell's proof of Theorem 4.7 uses simplicial decomposition, linear extension of a function to vertices, and the Tietze extension theorem. The arc A need not be the arc f(e)f(h), and the hypothesis does not require that f be bounded. For any locally bounded function f defined on the entire plane obtained by an extension of a continuous function on M in the manner described earlier, the function is bounded. Its range is a subset of M which is bounded.

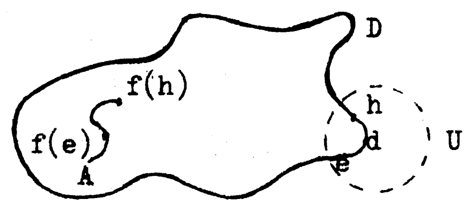


Figure 13. Illustration of Theorem 4.7

The importance of Theorem 4.7 lies in its use in the proof of Theorem 4.8, a theorem of major importance in the proof of Theorem 2.14.

<u>Theorem 4.8.</u> Let Q be a plane continuum and let g be a locally bounded function defined on the entire plane that is continuous at each point of Q and is such that $g(x) \in T(Q)$. Let $B = \{ x/\ x \notin T(Q)$ and $x \in T[Q \cup g_c(x)]\}$. Then if D is a simple closed curve for which $Q \subset T(D)$, either g_c has a fixed-point in $T(D)$ or $D \cap B \neq \emptyset$.

<u>Proof</u>: Suppose $D \cap B = \emptyset$. Let f be the restriction of g to $T(D)$. As was discussed earlier, the function f is bounded. The hypothesis implies $f(D) \subset T(Q) \subset T(D)$. To use Theorem 4.7 it must be shown that for each $d \in D$ there exists an open set U containing d such that if x,y are in $D \cap U$ then there is an arc A contained in $T(D)$ joining $f(x)$

to $f(y)$ such that $T(A \cup [f(x)f(y)]) \cap U = \emptyset$.

Suppose $d \in Q$. Let V be a spherical neighborhood about $f(d)$. Since $f(d) \in T(D)$, the topology of the plane implies there exists a V such that $V \cap T(D)$ is connected. Since f is continuous at each point of M there exists an open set U' such that $d \in U'$ and $f(U') \subset V$. The regularity of the plane implies there exist open sets A and B, $A \cap B = \emptyset$, $\overline{V} \subset A$, and $d \in B$. Let $U = B \cap U'$. The set U is open and $f(U) \subset V$.

Let $\{x,y\} \subset U \cap D$. Then $\{f(x)f(y)\} \subset V$. By Theorems 3.17 and 3.20, $V \cap T(D)$ is the interior of a topological two-cell. Therefore, there exists an arc A in $V \cap T(D)$ such that A joins $f(x)$ to $f(y)$ and $T(A \cup [f(x)f(y)]) \cap U \subset V \cap U = \emptyset$ as desired.

Suppose $d \notin Q$. By the assumption, $d \notin B$. Since $T[Q \cup f_c(d)]$ is bounded and does not separate the plane, there exists a ray L with end point d such that $L \cap T[Q \cup f_c(d)] = \emptyset$. By Lemma 4.2, the set $f_c(d)$ is nonempty and closed. Since L and $T[Q \cup f_c(d)]$ are closed, the set $L \cap T[Q \cup f_c(d)] = \emptyset$ and the plane is normal, there exist open sets U and V such that $L \subset U$, $f_c(d) \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. The set U can be chosen so that $C[f(U)] \subset V$ since $f_c(d) \subset V$. By the choice of U and V, $\overline{V} \cap L = \emptyset$ and $\overline{U} \cap T(Q) = \emptyset$.

Since V is an open set containing $T(Q)$, Theorem 4.4 implies there exists a simple closed curve J such that $Q \subset T(Q) \subset T(J)$ and $T(J) \subset V$. That is, $T(J) \cap (L \cup U) = \emptyset$. Let $\{x,y\} \subset U \cap D$. The set $\{f(x), f)y)\} \subset T(Q)$ where $T(Q) \subset T(D) \cap T(J)$. By Theorems 3.18 and 3.21, $T(D) \cap T(J)$ is a topological two-cell. Therefore, there exists an arc A joining $f(x)$ to $f(y)$, $A \subset T(D) \cap T(J)$ where $U \cap T(J) = \emptyset$. This completes the proof.//

Theorem 4.8 is one of the key theorems in Bell's paper. If one assumes g_c has no fixed-point in $T(D_n)$ for some n, the fact that $D_n \cap B \neq \emptyset$ must follow. Thus the set K such that $K \cap B \neq \emptyset$ for $K \in K_n$ is not empty.

The next theorem, Theorem 4.9, is also a key theorem. It states a sufficient condition for a continuum to be indecomposable. Once this condition is established, all the prerequisite theorems required for the proof of Theorem 2.14 will have been established. The purpose of the remaining theorems in the exposition is to demonstrate that Q fulfills the requirements stated in the hypothesis of Theorem 4.9.

It can be shown that the X in Theorem 4.9 must be a bounded continuum. However, since the theorem will only be applied in the case where X is a bounded continuum these conditions will be included as part of the hypothesis.

Theorem 4.9. Let h be a homeomorphism of the set of real numbers R into the plane for which $\lim_{x \to -\infty} |h(x)| = \infty$ and $h([0, \infty))$ is bounded. Let X be a bounded continuum such that $X = \overline{h(R)} - h(R)$. If for each $x \in X$ and each $e > 0$ there is a component C of $\{z/ \ |z-x| = e\} - T(X)$ and a real number r for which $h(t) \in T(C \cup X)$ for $t > r$, then X is indecomposable.

Proof: If $X = \emptyset$ is indecomposable by definition. If $X \neq \emptyset$, suppose X is decomposable; that is, $X = A \cup B$ where A and B are each proper subcontinua of X. Since $B \neq X$ and $A \cup B = X$, $A - B \neq \emptyset$. Let $a \in A - B$. Since B is closed and the plane is normal, there exist open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$, $a \in U$ and $B \subset V$. Since U contains an open connected set S containing a such that $\overline{S} \cap B = \emptyset$, let U be such a neighborhood.

Since $X = \overline{h(R)} - h(R)$, the point a is a limit point of $h(R)$ and there exists a real number v for which $h(v) \in U$. Since $A \neq X$ and $A \cup B = X$, $B - A \neq \emptyset$. Let $b \in B - A$. Because $h((-\infty, v)$ has $\lim_{x \to \infty} |h(x)| = \infty$, the set $h((-\infty, v]))$ is closed. Thus $\overline{U} \cup h((-\infty, v]) \cup A = H$ is closed. By the property of normality there exist open sets V and W containing b and H, respectively, such that $\overline{V} \cap \overline{W} \neq \emptyset$. Hence, there exists an $e > 0$ such that $Z = \{z/ \mid z - b \mid = e\} \subset V$ and has $Z \cap H = \emptyset$.

The hypothesis asserts that there is a component C of $Z - T(X)$ and a real number r for which $h(t) \in T(C \cup X)$ for $t > r$. Since $C \subset Z - T(X)$ C is a point or an open arc. If C is a point then for every open set W such that $C \subset W$, $W \cap T(X)$ can be shown to be nonempty from which it follows that C is a limit point of the closed set $T(X)$. This will now be proven.

Since W is open, W contains an open set $E = \{ x/ \mid x - C \mid = \varepsilon\}$ for some $\varepsilon > 0$. For θ such that $e\theta < \varepsilon/10$ there exists a point $z \in Z \cap E$ by the geometry of the plane. If $W \cap T(X) = \emptyset$, then $z \in Z - T(X)$ and there is an arc $zC \subset W$ such that $zC \subset Z - T(X)$. This contradicts the component C being a point. Thus $W \cap T(X) \neq \emptyset$ or $z \in T(X)$, and C is a limit point of the set $T(X)$. But since $C \notin T(X)$, C must be an open arc.

By the choice of C, the end points p and q of C are in X. But the choice of e implies $\{p,q\} \subset B$. It will now be shown that $X \subset \overline{h((r,\infty))}$.

Suppose there exists an $x \in X$ such that $x \notin \overline{h((r, \infty))}$. Then there exists an open set U containing x such that $U \cap \overline{h((r, \infty))} = \emptyset$. Since $h((-\infty,r))$ is unbounded there exists a $z \in R$ such that for every $y < z$, $y \notin U$. The set $U - \overline{h((r,y))} = K$ is open. If K contains x,

$K \cap \overline{h((r,y))} \cup \overline{h((-\infty,y))} \cup \overline{h(r,\infty))} = \emptyset$. This implies $x \notin X$ which is a contradiction. Therefore, $x \in \overline{h((r,y))}$, or since h is a homeomorphism, $x = h(u)$ for some $u \in [r, y]$. This contradicts the definition of X. Hence, the original assumption is false, and for $x \in X$, $x \in \overline{h((r, \infty))}$.

Since C and B are each bounded, $T(C \cup B)$ exists and is closed. Thus, $h(r,\infty) \subset T(C \cup B)$ implies $\overline{h((r,\infty))} \subset \overline{T(C \cup B)} = T(C \cup B)$.

Let $U \cup h((-\infty, v\,]) = N$. The set N is connected since it is the union of two intersecting connected sets. The choice of U implies that $N \cap (C \cup B) \subset N \cap (B \cup Z) = \emptyset$. The unboundedness of N implies $N \not\subset T(C \cup B)$. The fact that $a \in A$ implies $a \in X$. Thus, the point $a \in T(C \cup B)$. Since $\partial[T(C \cup B)] \subset C \cup B \subset Z \cup B$, $N = [N \cap T(C \cup B)] \cup [\,N - T(C \cup B)\,]$ sep. But this contradicts N being connected. Therefore, Q is not decomposable.//

One final preliminary definition, that of accessible point, is needed.

<u>Definition 4.6.</u> A point x of a continuum Q will be called an <u>accessible point</u> of Q if there is a ray with end point x that intersects Q only at the point x.

Implementation

Having discussed somewhat generally $T(Q)$, a locally bounded function, $f_c(x)$, S_n, F_n, K_n, A_n, and D_n, it is now time to consider their implementation in the proof of Theorem 2.14.

> <u>Theorem 2.14.</u> Let M be a bounded plane continuum which does not separate the plane, and let g be a locally bounded function defined on the plane such that $g(x) \in M$. If

> g is continuous at each point of M, then either g_c has a fixed-point in M or there is an indecomposable continuum $Q \subset M$ such that $Q = g(Q)$.

Assume g_c has no fixed points in $T(M)$; that is, g has no fixed-points in M. Choose a subcontinuum Q of M as follows: if the $\partial(M)$ contains a proper subcontinuum W for which $g(W) \subset W$, let Q be a minimal such; if no such W exists, let Q be $\partial[T(M)]$.

If one such W exists, it can be shown that a minimal W will exist. Let $W' = \{\partial(M) - W / W \neq \partial(M), W \subset \partial(M), g(W) \subset W\}$. By the assumption, the set $W' \neq \emptyset$. Partially order W' by set containment. Let C' be a chain in W', then $\cup\{C / C \in C'\}$ is an upper bound for C' because the compactness of $\partial(M)$ implies $\cap\{W / [\partial(M) - W] \in C'\}$ is a non-empty closed set. By Zorn's Lemma, W' has a maximal element. This means there exists a minimum $W \subset \partial(M)$ such that $g(W) \subset W$.

The sets S_n, F_n, K_n, A_n and D_n will be considered defined for the Q just described. Let f be the lb extension of g to $T(Q)$ such that f is continuous at each point of Q and $f(x) \in T(Q)$. The assumption that g_c has no fixed-points in $T(M)$ implies f_c has no fixed-points in $T(Q) \subset T(M)$. By Theorem 4.2, the set $T(Q) = \cap A_n$.

Let C be the set of fixed-points of f_c. The set C is closed, for if x is a limit point of C there exists a sequence of points $\{x_n\}$ converging to x such that $x_n \in f_c(x_n)$. By Theorem 4.6, $x \in f_c(x)$; that is, x is a fixed-point of f_c.

Suppose $A_n \cap C \neq \emptyset$ for every n. If $A_n \cap C$ is a finite set, then since $T(Q) = \cap T(D_n) = \cap A_n$, it is true that for some $x \in C$, $x \in T(Q)$. This contradicts the assumption that f_c has no fixed-points in $T(Q)$. If $A_n \cap C$ is an infinite set, the compactness of $\bar{A}_1$ implies $A_n \cap C$ has a limit point, x. Since x is a limit point of A_n for every n and $A_n = T(D_n)$ is closed, the point $x \in T(Q) = \cap T(D_n)$. Since x is a limit point of C and C is closed, the point $x \in C$. Hence, f_c has a fixed-point in $T(Q)$. Because this is a contradiction, there must exist an n' such that $A_{n'}$ contains no fixed-points of f_c.

For the remainder of the chapter $K' = \{K/\ K \in \bigcup_{n=n'}^{\infty} K_n$ and $K \cap B = \emptyset\}$ and $K'' = \cup\{K/\ K \in K'\}$. The set $K'' \subset \cup K_n$ and, therefore, satisfies the hypothesis of Theorem 4.3 with n' a fixed natural number. The set K' will be shown to be nonempty in the course of proving Theorem 4.11.

The fact that $Q \cup K''$ is bounded implies $T(Q \cup K'')$ exists. Let $Y = T(Q \cup K'')$. To consider the accessible points of Y, Definition 4.6 requires that Y be a continuum. This follows from the definition of K'' and $T(Q \cup K'')$. Since $Y \cup D_n$ is also bounded, $T(Y \cup D_n)$ exists. Let J_n be the boundary of $T(Y \cup D_n)$ for $n \geq n'$. The set $J_n \cap Y$ will be denoted by H_n for $n \geq n'$. It will now be shown that for some $N \geq n'$, the set $H_N \neq \emptyset$.

Let $N \geq n'$ such that for some $K \in K_{N'}$, $K \in K'$. Then $K \subset T[Y \cup D_{N+i}]$ but $K \not\subset T(D_{N+i})$ for $i \geq 1$. Such a K exists, for if not $K \subset T(M)$. This means $K \subset B$ which contradicts the definition of $K \in K'$. Consequently, there exists a $y \in J_{N+i} \cap K$ such that $y \notin D_{N+i}$.

By Theorem 4.3 (iii), each J_n is a simple closed curve. Since $T(Q \cup K'') = Y$ and $T(T(Q \cup K'') \cup D_n) = T(Q \cup K'' \cup D_n) = T(Y \cup D_n) = T(J_n)$, part (i) of Theorem 4.3 implies $\bigcap_{n=n'}^{\infty} T(J_n) = T(Q \cup K'')$. If D is a simple closed curve for which $Q \subset T(D) \subset \mathrm{Int}(T(D_n))$ then by Theorem 4.8, the set $D \cap B \neq \emptyset$. The sets J_n and H_n are illustrated in Figure 14. It is assumed that f_c is defined so that $K \cap B = \emptyset$

for $K \subset H_n$. The set $T(D_n)$ is marked \\\\ , and the set $T(Y \cup D_n)$ is marked //.

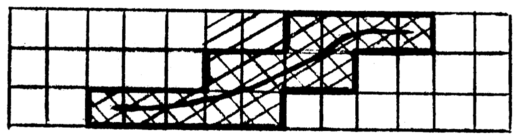


Figure 14. The Set $H_n = J_n \cap Y$

Conclusion of the Proof of Theorem 2.14

Before stating and proving theorems regarding H_n, Y and the accessible points, A(Y), of Y it is necessary to state an added theorem, Theorem 4.10.

<u>Theorem 4.10.</u> Let Q be a continuum and let N be a compact set. Let $x \in T(Q \cup N) - T(Q)$. Then there is a component K of N - Q such that $x \in T(K \cup Q)$.

This theorem will only be used in the proof of part (ii) of Theorem 4.11. Its proof, while relatively straight forward, is not included because it would not significantly contribute to an understanding of the proof of Theorem 2.14. Theorem 4.11, however, gives insight to the meaning of the sets H_n, Y and A(Y). Its proof clarifies these sets and their relationships. Since these ideas are essential in Bell's proof, Theorem 4.11 and its proof will now be presented.

<u>Theorem 4.11.</u> For H_n, K'' and n' as described,

i) $H_n \subset H_{n+1}$ for $n \geq n'$,

ii) if L is a ray that intersects $T(Q)$ only at its end point, x, then there is an open arc having x as one end point that is contained in both L and the interior of Y. Consequently, no accessible point of Y is in $T(Q)$.

iii) y is an accessible point of Y if and only if there exists an $n \geq n'$ for which $y \in H_n$,

iv) K'' contains the set of accessible points of Y: $A(Y) \subset K''$.

<u>Proof</u>: (i) The definitions of H_n and $T(D_{n+1} \cup Y)$ imply $H_n \subset Y \subset T(Y \cup D_{n+1})$. Since $H_n \subset \partial[T(Y \cup D_n)]$ and $T(Y \cup D_n) \supset T(Y \cup D_{n+1})$, the set H_n is in the $\partial[T(Y \cup D_{n+1})] = J_{n+1}$. Hence, $H_n \subset Y \cap J_{n+1} = H_{n+1}$.

(ii) Let L be a ray with end point x such that $L \cap T(Q) = x$. By Definition 4.6, x is an accessible point of $T(Q)$. Since $L \cap T(Q)$ is a single point and L is connected, the point $x \in \partial[T(Q)]$. By Lemma 4.1, $\partial[T(Q)] \subset Q$. Therefore, the point $x \in Q$.

The conditions on f imply f is continuous at $x \in Q$, and $f(x) \in T(Q)$. By Theorem 4.4, $f_c(x) = f(x)$. Since $f(x) \neq x$ and $L \cap T(Q) = x$, $f(x) \notin L$. Because the plane is regular there exists an open set V' containing $f(x)$ and having $V' \cap L = \emptyset$.

Let $\varepsilon = \rho(x, f(x))$. Let $N > n'$ such that $1/2^N < \varepsilon$. Let $S^* = \cup\{S \in S_N / x \in S\}$. Since $f_c(x)$ and S^* are disjoint closed sets, there exist open sets U and V'', $f_c(x) \subset V''$, $S^* \subset U$, $\overline{U} \cap \overline{V''} = \emptyset$. The set $V'' \cap V' = V$ is open and $V \cap (L \cup S^*) = \emptyset$.

From the definition of f_c, there exists an open set U' such that $x \in U'$ and $C(f(U')) \subset \overline{V}$. If $S^* \not\subset U' \cap U$ then for some $N' > N$, the set

$S' = \bigcup \{ S \in S_{N'} / x \in S \} \subset S^* \subset U' \cap U$. Since $S' \subset U'$, the set $C(f(S')$ $\subset \bar{V}$. Since $S' \subset U$, the set $S' \cap V = \emptyset$.

Let $W = \partial(S')$. Because L is unbounded $L \not\subset S'$. Let C be the component of $L - W$ such that $x \in C$. There exists a point $p \neq x$ such that $p \in C$. If no such p exists x is a limit point of the closed set W and $x \notin W$. This contradiction implies such a p exists. Therefore, $p \in T(Q \cup W)$ and $p \notin T(Q)$ as $L \cap T(Q) = \emptyset$. By Theorem 4.10, there exists a component N of $W - T(Q)$ such that $p \in T(Q \cup N)$.

Now $N \subset \{ K / K \in \cup K_n, K \subset T(W), K \cap N \neq \emptyset \}$. For such K, $f_c(K) \cap (K \cup N \cup L) = \emptyset$ by the previous choice of S'. Thus, such K cannot intersect B. The component N is therefore contained in Y and $T(Q \cup N)$ $\subset T(Q \cup K'') = Y$. The component C is in the interior of $T(Q \cup N)$ since $N \subset W$ and $C \not\subset Q \cup N$.

(iii) Let $y \in A(Y)$, the set of accessible points of Y. The definition of accessible point implies there exists a ray L with end point y such that $L \cap Y = y$. By (ii), the point $y \notin T(Q)$. Therefore, there exists an $n \geq n'$ for which $T(D_n) \cap L = \emptyset$. Since $y \in A(Y)$, the point $y \in \partial(Y)$. The set $Y = \bigcap_{n=n'}^{\infty} T(J_n)$. Therefore, there exists an $N \geq n \geq n'$ such that $y \notin \mathrm{Int}(T(J_N))$, that is $y \in J_N$. Thus $y \in (Y \cap J_N) = H_N$.

Let $y \in H_n$ for some $n \geq n'$. The definition of H_n implies y is an accessible point of Y.

(iv) Let $y \in A(Y)$. Then $y \in H_n$ for some $n \geq n'$. Since $H_n = J_n \cap Y$, $y \in \partial[T(Y) \cup D_n] \subset \partial(Y) \cup D_n$. Since $y \notin T(Q)$, there exists an $N \geq n$ such that $y \notin D_N$. Thus $y \in \partial(Y) = \partial[T(Q \cup K'')] \subset Q \cup K''$. But $h \notin T(Q)$ implies that $y \in K''$.//

Theorem 4.11 locates the accessible points of Y and prepares for the later use of $\overline{A(Y)} - A(Y)$ in the hypothesis of Theorem 4.9. The relationship between $\overline{A(Y)} - A(Y)$ and Q is not clear at this point. To provide some insight in this matter is one of the purposes of the next theorem.

<u>Theorem 4.12.</u> For Q, Y and A(Y) as defined

i) A(Y) contains no simple closed curve,

ii) A(Y) is open relative to $\overline{A(Y)}$. Furthermore, each component of A(Y) is homeomorphic to the set of real numbers,

iii) $\overline{A(Y)} - A(Y) \subset Q$.

<u>Proof:</u> (i) If A(Y) contained a simple closed curve D then $Q \subset T(D) \subset D_n$ by definition of D_n. Moreover, $B \cap D = \emptyset$ since by Theorem 4.11 (iv), $A(Y) \subset K''$. By Theorem 4.8, f_c has a fixed-point in T(D). This contradicts the assumption for f_c.

(ii) Let $y \in A(Y)$ and let $W = \cup\{K/\ K \in K' \text{ and } y \in K\}$. By definition, the set W is the union of open arcs with end points in T(Q). By Theorem 4.11 (iii) $y \in H_n$ for some $n \geq n'$. Since $y \in T(W \cup Q) - T(Q)$ there exists a D_n such that $y \notin T(D_n)$. These facts and the definition of H_n imply $y \in \partial(K'')$. Since $T(W \cup Q) \subset T(Y)$, y cannot be an interior point of $T(W \cup Q)$. Thus, $y \in \partial(W \cup Q)$.

Let $V = W \cap \partial[T(W \cup Q)]$. By the previous argument $y \in V$. Suppose $V = A \cup B$ sep. Since $K - T(Q) = K$, K is connected and there exist open arcs K_1 and K_2 contained in A and B, respectively, such that $y \in K_1 \cap K_2 \subset A \cap B$. Thus, V is connected and, by the definition of W, is an arc.

Since $y \notin T(Q)$, an open set U can be found that contains y and that intersects only those $K \in \cup K_n$ that contain y. This is true because there exists an $N \geq n'$ for which $1/2^N < \rho(y, T(D_N))$. Moreover, U can be chosen spherical and of small enough radius that $U \cap V$ is an open arc. Since $y \in \partial\left[T(W \cup Q)\right]$, the set U can be chosen so that $U \cap \left[P - T(Q \cup V)\right]$ is connected where P is the plane. That is $U \cap V$ is contained in the boundary of a square for the n such that for $N > n$, $K_N \cap y = \emptyset$. But this means $U \cap V \subset \partial(K'') \subset \partial\left[T(K'' \cup Q)\right] = \partial(Y)$. Therefore, $U \cap V \subset A(Y)$ and $A(Y)$ is open relative to $\overline{A(Y)}$.

Moreover, $U \cap V$ is an open arc. Since y was arbitrary in $A(Y)$, every point of $A(Y)$ has a neighborhood U such that $U \cap A(Y) = U \cap V$ is homeomorphic to an open interval. But this means each component of $A(Y)$ is homeomorphic to the reals.

(iii) Let $y \in \overline{A(Y)} - Q$. If $y \in A(Y) - Q$, the proof is finished. If $y \in \overline{A(Y)} - A(Y)$ and for every U containing y, $U \cap T(Q) \neq \emptyset$, then y is a boundary point of $T(Q)$. By Lemma 4.1, the $\partial\left[T(Q)\right] \subset Q$. Assume that for some U' such that $y \in U'$, the set $U' \cap T(Q) = \emptyset$. Then there exists an open set $U^* \subset U'$ such that $U^* \cap T(Q) = \emptyset$.

Since U^* intersects only a finite number of $K \in \bigcup_{n=n'}^{\infty} K_n$ and $A(Y) \subset K''$, y is in some $K \in K''$ such that K contains a point of $A(Y)$. The sets U and V can be defined as in (ii). Since $U \cap A(Y) \neq \emptyset$ and $y \in U \cap V \subset A(Y)$, y is in $A(Y)$.//

By Theorem 4.12 each component of $A(Y)$ is homeomorphic to the set of real numbers. Let L^* denote a component of $A(Y)$ containing the point y'. The homeomorphism just described indicates that $L^* - y' = L(y')$ is a component of $A(Y) - y'$ which is homeomorphic to $[0, \infty)$. Since $A(Y) \subset K''$, the set $A(Y)$ is bounded. Consequently, $L(y')$ is bounded.

The point y' being an accessible point of Y implies there exists a ray L' with end point y' such that $Y \cap L' = y'$. By definition

$K'' \subset Y$, therefore, $Y \cap L' \supset L' \cap K'' \supset L' \cap A(Y) = y'$. Now there exists a homeomorphism h of the set of real numbers onto $L = L' \cup L(y')$ for which $h((-\infty, 0]) = L'$, $h([0, \infty)) = L(y')$. Consequently, $\lim_{x \to -\infty} |h(x)| = \infty$ and $h([0, \infty))$ is bounded as is required in the hypothesis of Theorem 4.9. Another condition of Theorem 4.9 requires that Q equal $\overline{h(R)} - h(r) = \overline{L} - L$. From Theorem 4.12, the set $\overline{A(Y)} - A(Y) \subset Q$. But $\overline{L} - L$ is not exactly $\overline{A(Y)} - A(Y)$. Consequently, the first parts of Theorem 4.13 prove that $\overline{L} - L = Q$.

Part (ii) of Theorem 4.13 includes the statement that $f(\overline{L} - L) = \overline{L} - L$ or $f(Q) = Q$. Because of the choice of Q, this statement implies $\overline{L} - L$ must be Q.

In order to use Theorem 4.9 to prove that Q is indecomposable one other condition is necessary, namely, that for each $x \in Q$ and each $e > 0$ there exists a component C of $\{ z / \ |z - x| = e \} - T(Q)$ and a real number r for which $h(t) \in T(C \cup X)$ for $t > r$. This will be proved in (iv) of Theorem 4.13. Part (iii) will be used to prove part (iv).

Theorem 4.13 will now be stated and proved.

Theorem 4.13. For L, Q and f as defined

i) $\overline{L} - L$ is a subcontinuum of Q.

ii) $f(\overline{L} - L) \subset \overline{L} - L$. Hence, $\overline{L} - L = Q = f(\overline{L} - L)$.

iii) if A is a closed arc that intersects Q only at the end points of A then $h^{-1}(A)$ is bounded, and

iv) Q is indecomposable.

Proof: (i) Since $\overline{L} - L \subset \overline{A(Y)} - A(Y)$, the set $\overline{L} - L$ is bounded. By Theorem 4.12 (ii), for each point $y \in L$ there exists an open set U relative to $\overline{A(Y)}$ such that $U \cap \overline{A(Y)} - A(Y)) = \emptyset$. Therefore, for

$y \in L$, y is not a limit point of $\overline{L} - L$.

If y is a limit point of $\overline{L} - L$, then for any open set U containing y, the set $U \cap (\overline{L} - L) \neq \emptyset$. But this implies $U \cap L \neq \emptyset$ and that $y \in \overline{L} - L$. Thus, $\overline{L} - L$ is closed.

Because h is a homeomorphism and $\overline{h(R)} - h(R) = \overline{L} - L$, the argument given in the proof of Theorem 4.9 implies $\overline{L} - L \subset \overline{h([r, \infty))}$ for any real number r. The set $h([0, \infty))$ being connected indicates that $\overline{h([0, \infty))}$ is connected. It is also closed and bounded. Let $\{x_n\}$ be the sequence of integers. A similar argument forces $\overline{h([x_n, \infty))}$ to be a continuum for every n. By Newman (37) page 81, $\overline{h([x_n, \infty))} = \overline{L} - L$ is a continuum.

ii) Let $x \in \overline{L} - L \subset Q$ and suppose $f(x) \notin \overline{L} - L$. As in (ii) of Theorem 4.11, there exists an $N \geq n'$ for which $f_c(S') \cap (L \cup S') = \emptyset$. Since $x \in Q$ the $\mathrm{Int}(S')$ contains some point $z \in L(y')$.

Letting $W = \partial(S')$ makes W a compact set. Since $z \in \mathrm{Int}(S')$, $z \in T(W \cup Q)$. By Theorem 4.10 there exists a component C of $W - T(Q)$ such that $z \in T(Q \cup C)$. The set $C \cup L$ is connected and unbounded. The choice of N implies $f_c(T(W)) \cap (L \cup C) = \emptyset$. Thus, any $K \subset \bigcup_{n=n'}^{\infty} K_n$, $K \subset W$ and $K \cap C \neq \emptyset$ has $K \cap B = \emptyset$ by the definition of B. Therefore, $C \subset K''$. Because $z \in \mathrm{Int}(S)$, z is in the $\mathrm{Int}\,\left[T(C \cup Q)\right] \subset T(K'' \cup Q) = Y$. This is impossible since $z \in L(y') \subset A(Y)$ which forces $z \in \partial(Y)$.

Since $f(\overline{L} - L) \subset \overline{L} - L$ and Q was chosen so that it was a minimum continuum for which $f(Q) \subset Q$, the set $\overline{L} - L = Q = f(Q)$.

(iii) Since h is a homeomorphism, A an arc, and $h^{-1}(L)$ is unbounded, $h^{-1}(A)$ is bounded below. If $h^{-1}(A)$ is not bounded above then $A \cap L(y')$ has a limit point $z \in Q$, namely, one end point of A. Since A is an arc, $z \in A$, and there exists a ray F with end point z such that $F \subset L \cup A$, or equivalently, that intersects $T(Q)$ at precisely the point z. By Theorem 4.11 (ii) there exists an open arc having z as one end point that is contained in $F \cap \mathrm{Int}(Y)$. This contradicts z being a limit point of $L(y') \subset A(Y)$. Therefore, $h^{-1}(A)$ must be bounded above.

(iv) In order to use Theorem 4.9 and prove that Q is indecomposable, it is sufficient to show that for $e > 0$ and $x \in Q$ there exists a component D' of $D - T(Q) = \{ x / \mid z - x \mid = e \} - T(Q)$ and a real number r for which $h(t) \in T(Q \cup D')$ for any $t > r$.

As was argued earlier if $D - T(Q) \neq D$ then each component of $D - T(Q)$ is an arc with end points in Q. Using (ii) of this theorem, it is possible to define two sequences of real numbers $\{ x_i \}$ and $\{ y_i \}$.

Let x_1 be the smallest x such that $x \in h^{-1}(D)$. Such a point exists because L is unbounded and $L \cap \{z / \mid z - x \mid \leq e \} \neq \emptyset$. Let y_1 be the largest real number for which $h(x_1)$ is in the same component of $D - T(Q)$ as $h(y_1)$. In general if x_i has been defined let y_i be the largest number for which $h(x_i)$ is in the same component of $D - T(Q)$ as $h(y_i)$. If y_i has been defined let x_{i+1} be the smallest real number in $h^{-1}(D)$ that is greater than y_i. Such a number will exist because y_i and x_{i+1} will be in disjoint arcs each of which satisfies (ii) of this theorem.

Using the sequence $\{ x_i \}$ and $\{ y_i \}$ a homeomorphism h' can be defined as follows:

$$h'(x) = \begin{cases} h(x) \text{ if } x \notin \cup \{ x_i y_i / \ i = 1,2,\ldots \} \\ g(x) \text{ if } x \in \cup \{ x_i y_i / \ i = 1,2,\ldots \} \text{ where } g \text{ is a homeomorphism} \\ \text{mapping } x_i y_i \text{ onto the arc } h(x_i)h(y_i) \subset D - T(Q). \end{cases}$$

Since $h(x)$ and $g(x)$ agree at x_i and y_i, the properties of h and g imply h' is a homeomorphism.

Let L" be the image of h'. By an argument similar to that used in part (ii) it can be shown that $f(L'' - L'') \subset \overline{L''} - L''$. Hence $Q = \overline{L''} - L''$. The value of r may be any real number for which $| h'(r) - x | < e$. The component D' may be any component of $D - T(Q)$ for which $h(r) \in T(D' \cup Q)$.

Since e and x were arbitrary, it is true that for each $x \in X$ and each $e > 0$ there is a component of D' of $D - T(Q)$ and a real number r for which $h(t) \in T(Q \cup D')$ for any $t > r$. By Theorem 4.9, Q is indecomposable.//

Dependent Results

As was mentioned in Chapter II, Charles Hagopian used Theorem 2.14 to prove Theorem 2.19.

> Theorem 2.19. If M is an arcwise connected bounded continuum which does not separate the plane, then M has the fixed-point property.

In order to show that the boundary of the continuum M in Theorem 2.19 does not contain an indecomposable continuum, Hagopian (16) needed Theorem 4.14. He stated and proved this theorem, along with Theorem 2.19, in "A Fixed Point Theorem for Plane Continua" in the Bulletin of American Mathematical Society in 1971.

Theorem 4.14. Suppose M is a continuum in the plane P, P - M does not have infinitely many components, and $\partial(M)$ contains an indecomposable continuum I. Then every subcontinuum of M which contains a nonempty open subset of I must contain I.

The proof of Theorem 2.19, in its entirety, is as follows.

Theorem 2.19.

Proof: If $\partial(M)$ does not contain an indecomposable continuum then by Bell's Theorem 2.14, the proof is complete. Suppose I is an indecomposable continuum in $\partial(M)$. Let q be a point of M - I. Let p be a point of I and let $\{U_n\}$ be a monotone sequence of circular regions in P centered on and converging to p. Since M is arcwise connected, for each point x of $I - U_n$ there exists an arc xq. The indecomposable continuum I is not a subset of xq because xq is hereditarily decomposable.

For each $n \in I^+$, let H_n be the set of points of I that can be joined with q by an arc in $M - U_n$. The properties of the plane assure that for every x in I - p, there exists an n such that $x \in H_n$. Consequently, $I \subset (\cup H_n) \cup p$. Since $\cup H_n \cup p$ is contained in I by construction, $I = (\cup H_n) \cup p$. If $\overline{H}_n$ does not contain a nonempty subset of I, then I is a countable union of nowhere dense subsets of I. This contradicts the Baire Category Theorem. Therefore, for some j, the set $\overline{H}_j$ contains a nonempty open subset of I. Clearly, $\overline{H}_j \cup \{xq/\ x \in H_j\}$ is a continuum in M. By Theorem 4.14, $I \subset \overline{H}_j \cup \{xq/\ x \in H_j\}$. But this is impossible since $p \notin \overline{H}_j$ for any j. This contradiction implies the $\partial(M)$ does not contain an indecomposable continuum. By Bell's Theorem 2.14, M has the fpp.//

At present Theorem 2.19 is the only major result using Theorem 2.14. However, as was mentioned in the conclusion of Chapter II, O. H. Hamilton is working on applying Bell's Theorem to obtain a complete answer to the fixed-point question. Hopefully, his result will be forthcoming soon.

CHAPTER V

SUMMARY

Prior to World War I, the question "Does an arbitrary bounded plane continuum which does not separate the plane have the fixed-point property?" began to intrigue mathematicians. In those early years the question was asked in conversation, but it did not appear in print. Gradually, however, partial answers to the question were obtained and these, along with the question itself, began to appear in various articles.

The first general result was disclosed in 1932 when K. Borsuk (8) proved that all non-separating plane Peano continua have the fixed-point property. The mathematical evolution culminating in Borsuk's paper is made easily accessible in Chapter II.

After 1932, the next significant contribution to fixed-point results was made by O. H. Hamilton (22) in 1938. His use of indecomposable continua introduced a changing approach toward fixed-point question. The importance of this approach became evident in 1967 with the publication of H. Bell's (3) result that a non-separating plane continua either has a fixed-point or contains an indecomposable continuum.

The importance of Bell's paper as well as its relationship to other fixed-point results is discussed in Chapter II. An effort to expose Bell's work occupies the major portion of Chapter IV. Because of its complexity, only a handful of mathematicians have read and attempted to comprehend Bell's paper. Against this background, it is understandable that the author of this thesis feels the exposition in Chapter IV is not completely adequate. It is, however, the only known effort to improve the readability of Bell's paper and to make it accessible to graduate students in mathematics.

In 1951, O. H. Hamilton (20) proved that snake-like continua have the fixed-point property. This proof is a highly significant contribution because it completely answers the fixed-point question for snake-like continua. It is also an amazing accomplishment on the part of Hamilton because no lesser theorems seem to have been published prior to his general result.

In 1971, C. Hagopian (16) proved that all arcwise connected continua which do not separate the plane have the fixed-point property. A number of proofs for special classes of arcwise connected continua preceded Hagopian's general result. The interdependence in these proofs is discussed in Chapter II.

The compilation of partial "fixed-point question" solutions along with examples and clarifications, is one contribution of this thesis. More important, however, is the exposition of the intertwining evolution of the fixed-point results. This exposition includes present efforts, and is located in Chapter II.

In addition, the proofs of major fixed-point theorems were analyzed. This analyzation resulted in the identification of five fundamental techniques. The names attached to these techniques attempt to capture the outstanding aspect of the technique. The names are: "dog-chases-rabbit", "immediate", "cyclic element", "change of topology", and "sequence of arcs."

The "dog-chases-rabbit" technique requires a sequence of points $\{x_n\}$ such that for every $\varepsilon > 0$ there exists an n with $\rho(x_n, f(x_n)) < \varepsilon$. It is the oldest, most frequently used, and best known approach to fixed-point proofs.

The "immediate" technique was so named because it involves a function from which the result is easily observed. In order for the cyclic element technique to be applicable the continuum must be a Peano continuum. The usefulness of this approach was exhausted with Borsuk's proof that non-separating plane Peano continua have the fixed-point property.

In a "change of topology" proof one uses a collection of subsets, other than the original, as a basis for a new topology. For a "sequence of arcs" proof, every monotone increasing sequence of arcs in the continuum must be contained in an arc. These two techniques have not been used extensively in obtaining solutions to the fixed-point question.

The identification, naming, and characterization of these techniques is one of the contributions of Chapter III. The major contribution of this chapter, however, is its analyzation of fixed-point proofs relative to the five fundamental techniques.

At the completion of this thesis the fixed-point question is still unanswered, in general, for nonarcwise connected continua. Thus, the question "Does an arbitrary bounded plane continuum which does not separate have the fixed-point property?" continues to tantalize contemporary mathematicians.

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APPENDIX A

PROOFS

1. Theorem 2.1. A homeomorphic mapping of a dendrite into itself has at least one fixed-point.

Proof: Let D denote the dendrite and f be a homeomorphism such that f maps D into D. To attain the proof assume f leaves no point fixed. Let p_1 be a point of D and let $p_2 = f(p_1)$. Since D is a dendrite there exists a unique arc p_1p_2 from p_1 to p_2. The image of p_1p_2 under the homeomorphism f is an arc p_2p_3 where $f(p_2) = p_3$. Since the arc joining any two points of a dendrite is unique, one of the following three cases must occur:

Case I. The arc $p_2p_3 \subset p_1p_2$ or $p_1p_2 \subset p_2p_3$.

If $p_2p_3 \subset p_1p_2$, f maps a homeomorphic image of an interval onto a subcontinuum and the Brouwer Fixed Point Theorem, 1.1, implies that f has a fixed-point. If $p_1p_2 \subset p_2p_3$, a similar argument applies to f^{-1}. Since f is a homeomorphism, f^{-1} having a fixed-point implies f has a fixed-point.

Case II. The set $p_1p_2 \cap p_2p_3 = p_2$.

The set $p_2p_3 \cap p_3p_4 = p_3$ where $f(p_2p_3) = p_3p_4$. For suppose there exists a point c such that $c \neq p_3$ and $c \in p_2p_3 \cap p_3p_4$. Since $c \in p_3p_4$ and $c \neq p_3$ then there exists a point $b \in p_2p_3$ such that $b \neq p_2$ and $f(b) = c$. Since $c \in p_2p_3$ and $c \neq p_3$ there exists a point $a \in p_1p_2$ such that $f(a) = c$. Therefore, $f^{-1}(c)$ is not unique. This contradicts the fact that f is a homeomorphism. Therefore, $p_2p_3 \cap p_3p_4 = p_3$ as was desired.

The set $p_1p_2 \cap p_3p_4 = \emptyset$. Suppose there exists a point of $c \in p_1p_2 \cap p_3p_4$. Since $c \in p_3p_4$, the properties of an arc imply there exist arcs $p_3c \subset p_3p_4$ and $cp_2 \subset p_1p_2$. Since $p_1p_2 \cap p_2p_3 = p_2$, $cp_2 \cup p_2p_3$ unites c with p_3 and is an arc. But $p_3c \subset p_3p_4$ and $p_2p_3 \cap p_3p_4 = p_3$ imply $cp_2 \cup p_2p_3 \neq p_3c$. This gives two arcs joining p_3 and c which is the desired contradiction.

In a similar way it can be argued that the infinite sequence of arcs p_1p_2, $p_2p_3, \ldots, p_np_{n+1} = f(p_{n-1}p_n)$, has the property $p_{n-1}p_n \cap p_np_{n+1} = p_n$. Since a dendrite is a hereditarily locally connected continuum, an arc minus its end points is a connected open set, and $\delta(p_np_{n+1}) = \delta(p_np_{n+1} - \{p_n, p_{n+1}\})$. Lemma 1 implies that $\lim_{n \to \infty} \delta(p_np_{n+1}) = 0$.

The following argument proves that $\{p_n\}$ converges to a single limit point. Since D is compact, some subsequence of $\{p_n\}$ must converge. Suppose two subsequences converge such that there exist two distinct limit points, x and y. Let U be a neighborhood of x such that $\delta(U) < \rho(x,y)/4$ and let V be a neighborhood of y such that $\delta(V) < \rho(x,y)/4$. Local connectivity implies there exists a neighborhood W of x such that $W \subset U$ and W is connected with respect to D. Since x is a limit point there exist infinitely many p_n in W. Pick one, say p_i. Since y is also a limit point, there exists a $j > i$ such that $p_j \in V$. The construction of the p_np_{n+1} and the definition of arc implies that $\bigcup_{n=i}^{j} p_np_{n+1}$ is an arc joining p_i and p_j. Similarly there exists a $k > j$ such that $p_k \in W$ and $\bigcup_{n=i}^{k} p_np_{n+1}$ is an arc joining p_i and p_k. W connected with respect to D means there exists an arc joining p_i and p_k

which lies entirely in W and, therefore, has diameter less than $\rho(x,y)/4$. The diameter of $\bigcup_{n=i}^{k} p_np_{n+1}$ is necessarily greater than $\rho(x,y)/4$ because of the choice of diameters for U and V. This means two arcs join p_i and p_k. This cannot happen in a dendrite. Therefore, the $\{p_n\}$ can converge to at most one point, p. The function f being continuous implies that $\{f(p_n)\}$ converges to f(p). Therefore, for every $\varepsilon > 0$, n can be chosen such that $\rho(p, f(p)) < \rho(p, p_n) + \rho(p_n, f(p_n)) + \rho(f(p_n), f(p)) < \varepsilon$ and $f(p) = p$.

Case III. The set $p_1p_2 \cap p_2p_3$ is an arc p_2p.

Let x be in D and let x vary along the arc from p_1 toward p. At each point x in p_1p, $x \neq p_1$, the set p_1x is an arc contained in p_1p by Theorem 3.3. Since f is a homeomorphism, $f(p_1x)$ is an arc $p_2f(x)$. For x in p_1p, f(x) is in p_2p_3 because $f(p_1p_2) = p_2p_3$. For x = p, the point f(p) either belongs to pp_2 or it belongs to pp_3.

Suppose f(p) is in pp_2. A sequence of points $\{x_n\}$ such that for every n, $x_n < x_{n+1} < f(x_{n+1}) < f(x_n)$ on $pf(p_n)$, will be defined inductively.

Let $x_1 = p$. Clearly f(p) is in pf(p) and f(p) > p on pf(p). By the definition of arc there exist points x in pf(p) such that $x \neq p$ and $x \neq f(p)$. For any such point x, x follows p on pf(p) by Definition 3.1.

Claim 1: For every x in pf(p), the point f(x) is in pf(p).

Since $pf(p) \subset p_1p_2$, the set $f(pf(p)) \subset p_2p_3$ and f(x) is in p_2p_3. By Theorem 3.2, the arc $p_2p_3 = [p_2f(p)] \cup [f(p)p] \cup [pp_3]$. Suppose $f(x) = z$ is in p_2p. In this case z has two pre-images, namely, x and some point b in p_1p where $b \neq x$ since $x \notin p_1p$. This cannot happen because f is a homeomorphism.

Suppose f(x) is in pp_3; that is, f(x) > p on pp_3. Then f(px) =

$= f(p)f(x)$. Since $f(x) > p$ on pp_3, $f(x) > p$ on $\left[f(p)p\right] \cup \left[pf(x)\right] =$ $f(p)f(x)$ by Theorem 3.2. But $\{p,x\} \subset f(p)f(x)$ since $\{p,x\} \subset f(p)p$. By Theorem 3.3 px is a subarc of $f(p)p$. But $f(px) = f(p)f(x)$. Therefore f^{-1} maps $f(p)f(x)$ onto the subarc px and f^{-1} has a fixed-point by Theorem 1.1. This means f has a fixed-point since f is a homeomorphism. This is a contradiction of the original assumption. Therefore, $f(x)$ is in $f(p)p$ as desired.

Claim 2: For at least one such x, $x \neq p$, $f(x) > x$ on $pf(p)$.

By Theorem 3.1 if $x \nless f(x)$ then $f(x) < x$ or $f(x) = x$. Since f is assumed to be fixed-point free, $f(x)$ cannot equal x. So, suppose $f(x) < x$. For some $\varepsilon > 0$ there exist open sets U and V with p in U and $f(p)$ in V such that $U \cap V = \emptyset$. Now any open set W containing p contains an open set T such that $T \subset U$ and T contains a point x in $pf(p)$, $x \neq p$, $x \neq f(p)$. If $f(x) < x$ on $pf(p)$, then $f(x)$ is not in V and the continuity of f is contradicted. Therefore, there exists at least one x, $x \neq p$, such that $f(x) > x$ on $pf(p)$.

Pick one such x and call it x_2. Then $x_1 < x_2 < f(x_2) < f(x_1) = f(p)$ on $pf(p)$.

Assume x_n has been defined such that for every $n \in I^+$ the point x_n is in $pf(p)$ and $x_{n-1} < x_n < f(x_n) < f(x_{n-1})$ on $pf(p)$. Then x_{n+1} can be defined so that $x_{n+1} \in pf(p)$ and $x_n < x_{n+1} < f(x_{n+1}) < f(x_n)$ on $pf(p)$.

As in the preceding argument, there exist points x in $x_n f(x_n) \subset pf(p)$ such that $x_n \neq x$, $x \neq f(p)$, $f(x) \in pf(x_n)$. Substituting x_n for p and $f(x_n)$ for $f(p)$ in the proof of Claim 2, it can be shown that there exists at least one x in $x_n f(x_n)$ such that $f(x) > x$. Pick one such x and call it x_{n+1}. Then x_{n+1} has been defined in such a way that it satisfies the desired conditions.

By the induction argument just given, $x_{n+1} > x_n$. Thus, a sequence of distinct points $\{x_n\}$ has been defined. The compactness of D implies that some subsequence $\{x_k\}$ of $\{x_n\}$ must converge to a point x_o. Without loss of generality assume $\{x_n\}$ converges to x_o. The continuity of f implies that $\{f(x_n)\}$ converges to $f(x_o)$. From the definition of a metric, $\rho(x_o, f(x_o)) = 0$ or $\rho(x_o, f(x_o)) > 0$. If $\rho(x_o, f(x_o)) = 0$, then $f(x_o) = x_o$ and f has a fixed-point.

If $\rho(x_o, f(x_o)) > 0$, then there exists an $r > 0$ such that $\rho(x_o, f(x_o)) = r$. By the induction argument and Theorem 3.3 the arc $x_of(x_o) \subset x_nf(x_n)$ for every n. Since $x_of(x_o) \subset pf(p)$, the set $f(x_of(x_o)) = f(x_o)f^2(x_o) \subset pf(p)$ by the argument in Claim 1. Suppose $f^2(x_o)$ is in the arc $f(p)f(x_o)$. Then $f^2(x_o)$ has two pre-images; one in p_1x_o and $f(x_o)$ which is not in p_1x_o. This contradicts the one-to-oneness of f. Therefore, $f^2(x_o)$ is in $pf(x_o)$ and Case 1 is invoked.

If f(p) is in pp_3, then by Theorem 3.2 p is in $p_2p \cup pf(p) = p_2f(p)$. Since $f(p_1p) = p_2f(p)$, there exists a point b in p_1p such that $f(b) = p$. By Theorem 3.3 $bp \subset p_1p$. The set $f(bp) = f(b)f(p) = pf(p)$ which is a subset of pp_3 by Theorem 3.3 since f(p) is in pp_3. Therefore, $pf(p) \cap bp \subset pp_3 \cap p_1p = p$ and the use of Case 2 concludes the proof.//

2. Theorem 2.2. In order that a Peano continuum have the fixed-point property it is necessary that it be unicoherent.

Proof: Suppose the opposite. Let M be a Peano continuum which is not unicoherent. Then by Theorem 3.6 there exist Peano continua A and B such that $M = A \cup B$ where $A \cap B$ is not a continuum. Since A and B are closed, $A \cap B$ is closed and compact. Consequently, there exist sets P and Q such that $A \cap B = P \cup Q$ sep. Moreover, P and Q must each be closed for if not, the separation of P and Q implies $P \cup Q$ is not closed.

Since $P \cup Q$ sep, there exist p in P and q in Q, $p \neq q$. The fact that A and B are Peano continua assures that for each $a \in A$ and $b \in B$ there are arcs $ap \subset A$ and $bq \subset B$.

By Theorem 3.7, there exists a continuous function f defined on A such that $f(A) = bp$, $f(P) = q$ and $f(Q) = p$. Similarly, there exists a continuous function g such that $g(B) = ap$, $g(P) = q$, and $g(Q) = p$. Extend f to $A \cup B$ by letting $f = g$ on B. Because $A \cap B = P \cup Q$ and $f = g$ on $P \cup Q$, the function f is well-defined and continuous on $A \cup B$ with image $ap \cup bq$.

The proof is completed by showing that f has no fixed-point. If f has a fixed-point x, it can be assumed without loss of generality that x is in A. Thus $f(x)$ is in $bq \subset B$. But $f(x) = x$ implies $f(x)$ is also in A. Consequently, $f(x) = x$ is in $A \cap B = P \cup Q$.

If $x \in P$ the definition of A implies $f(x) = q$ which is in Q. This means that $x \in Q$ which is a contradiction. A similar contradiction is obtained if $x \in Q$. Thus, f has no fixed-point.//

3. <u>Theorem 2.6.</u> If D is a bounded simply connected domain in the plane which together with its boundary, does not separate the plane and whose outer boundary M contains no indecomposable continuum, then every homeomorphism of $\overline{D}$ into itself leaves some point of $\overline{D}$ invariant.

<u>Proof:</u> The bounded complementary domain G of the outer boundary M of D is itself a simply connected domain since it is D union the inner

boundary points of D by Lemma 2. The set G is a subset of $\overline{D}$ and has boundary M. The domain G, therefore satisfies the hypothesis of the theorem, and any homeomorphism on $\overline{D}$ is a homeomorphism on $\overline{G}$. Without loss of generality we may assume the boundary of D to be identical with its outer boundary.

From Collingwood and Lohwater (13) page 173, there exists an analytic homeomorphism g of the interior I of a given circle J into D such that $g\colon I \cup J \to D^*$ and g_J is one-to-one and onto the set of all prime ends of D*. This correspondence behaves as follows: if $\{q_i\}$ is a sequence of points of I converging to a point q of J, then the sequence of points $\{g(q_i)\}$ converges to the prime end $Q = g(q)$, and if $\{p_i\}$ is a sequence of points of D converging to a prime end P of D*, then $\{g^{-1}(p_i)\}$ converges to the point p of J where $g^{-1}(P) = p$.

Let f be a homeomorphism of $\overline{D}$ into itself. Since the points of D do not change from $\overline{D}$ to D* and f is an open mapping, $f_{1D} = f_D$ is well-defined.

Let P be a prime end defined by a chain $\{q_n\}$. The properties of f and D imply $\{f(q_n)\}$ is a chain. Let $f_1(P) = Q$ where Q is the prime end defined by $\{f(q_n)\}$. Since f is a homeomorphism, the function f_1 is also a homeomorphism.

Define a transformation h on $I \cup J$ into itself. If p is in I, let $h(p) = g^{-1}\left[f_1(g(p))\right]$. If p is in J, let $\{p_i\}$ be a sequence of points of I converging to p. Let P be the prime end of D* associated with p by g. The sequence $\{g(p_i)\}$ converges to P. Since f_1 is a homeomorphism on D*, the sequence $\{f_1(g(p_i))\}$ converges to the prime end $f_1(P)$. The properties of g^{-1} imply $g^{-1}\left[f_1(g(p_i))\right]$ converges to a point of J. Call that point h(p). Thus h is defined for all p in

$J \cup I$ and $h(I \cup J) \subset I \cup J$.

The mapping h is one-to-one and onto since each of the composition functions is one-to-one and onto. One can show h is a homeomorphism by showing that for every $A \subset I \cup J$, $h(\bar{A}) = \overline{h(A)}$.

Let A be a subset of I and $\{p_i\}$ a sequence of points of A converging to p. By the definition of h, the sequence $\{h(p_i)\}$ converges to h(p) and $h(\bar{A}) \subset \overline{h(A)}$. Similarly for $\{q_i\}$ in h(A) converging to q, $\{h^{-1}(q_i)\}$ converges to h(q) and $\overline{h(A)} \subset h(\bar{A})$. Therefore, $h(\bar{A}) = \overline{h(A)}$ for every $A \subset I$.

Let A be a subset of J and $\{q_i\}$ a sequence of A converging to a point q. Since J is a continuum, the point q is in J. In order to show that $\{h(q_i)\}$ converges to h(q), it must be shown that for each positive integer n there exists a point z_n of I such that $\rho(z_n, q_n)$ and $\rho\left[h(z_n), h(q_n)\right]$ are each less than 1/n.

Since q_n is in the boundary of I, there exists a sequence of points $\{z_{n_i}\}$ in I converging to q_n for every n. The definition of h implies $\{h(z_{n_i})\}$ converges to $h(q_n)$. The definition of convergence implies there exist integers k and j such that $\rho(z_{n_k}, q_n) < 1/n$ and $\rho\,[h(z_{n_j}), h(q_n)] < 1/n$. Then for $L = \max\{k, j\}$, z_{n_L} is the desired z_n.

Let $\varepsilon > 0$. For every $\varepsilon > 0$, there exists $1/n < \varepsilon/2$ and, consequently, there exists an n such that $\rho(z_n, q) \leq \rho\,(z_n\ q_n) + \rho(q_n, q) < 1/n + 1/n < \varepsilon$. Thus $\{z_n\}$ converges to q and $\{h(z_n)\}$ converges to h(q). Consequently for $\varepsilon > 0$ and $1/n < \varepsilon/2$, $\rho\left[h(q_n), h(q)\right] \leq \rho[h(q_n), h(z_n)] + \rho\,[h(z_n), h(q)] < 1/n + 1/n < \varepsilon$ and $\{h(q_i)\}$ converges to h(q). Thus, $h(\bar{A}) \subset \overline{h(A)}$. Similarly $\overline{h(A)} \subset h(\bar{A})$. Therefore, $h(\bar{A}) = \overline{h(A)}$ for every $A \subset J$.

Let A be a subset of $I \cup J$. If $\{q_i\}$ is a convergent sequence of A then $\{q_i\}$ is either a sequence contained in I except for a finite number of points or a sequence contained in J except for a finite number of points, or for every N there exist n and m greater than N such that q_n is in J and q_m is in I. Since $I \cup J$ is a metric space, if $\{q_n\}$ converges to a and $\{q_m\}$ converges to b, a = b. Therefore, the preceding argument implies $h(\overline{A}) = \overline{h(A)}$. Thus $h(\overline{A}) = \overline{h(A)}$ for every A in $I \cup J$ and h is a homeomorphism on $I \cup J$.

By Theorem 1.1, the homeomorphism h leaves some point of $I \cup J$ fixed. If h leaves a point of I fixed then $h(p) = g^{-1}[f_1(g(p))] = g^{-1}f_1(q) = g^{-1}(z) = p$. Since g is a homeomorphism and $g^{-1}(z) = p$, it follows that $z = g(p)$ or $z = q$. Therefore, $f_1(q) = q$ anf f_1 leaves a point of D fixed. Since f and f_1 agree on D, tne function f leaves a point of $\overline{D}$ fixed.

Suppose h leaves a point p of J fixed. A similar argument shows f_1 carries some prime end P of D* into itself. Let $N = \cap \overline{d}_n$ where d_n is in the original topology for $\overline{D}$ and is associated with P. Let q be an arbitrary point in N. There exists a sequence of points $\{q_n\}$ such that q_n is in d_n and $\{q_n\}$ converges to q. Consequently, $f(q_n)$ is in $f(d_n)$ for every n and $\{f(q_n)\}$ converges to f(q). Since $f(q_n)$ is in $f(d_n)$ for every n, and $f(d_n) \supset f(d_{n+1})$ for every $i \geq 1$, f(q) is in $f(\overline{d}_n)$ for every n. But $f_1(P) = Q$ and the definition of f_1 imply $\cap f(\overline{d_n}) = \cap \overline{d_n} = N$. Therefore, f(q) is in N and f maps N into itself.

Theorem 3.9 gives a necessary condition that N shall be the whole boundary of D, namely, the boundary of D must be indecomposable or the sum of two indecomposable subcontinua. Since by the hypothesis the boundary of D contains no indecomposable continuum, N is a proper subcontinuum of the boundary of C. Let E represent the plane. The set $E - \overline{D} = S$ is connected and $E - \overline{S} = D$ is connected. Therefore, $E - N = S \cup D \cup (\partial(D) - N)$ is connected and N is a compact continuum which does not separate the plane. Since $N \subset \partial(D)$, the continuum N contains no domain. By Theorem 2.5, f leaves some point of N fixed. It follows that in any case f leaves a point of $\overline{D}$ fixed.//

4. <u>Theorem 2.7.</u> Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fixed-point property.

<u>Proof</u>: The hypothesis implies M contains no simple closed curve in either its original topology or in its arc-topology. By Theorem 3.28, M is Hausdorff in the arc-topology.

To show that M is arc-locally connected it is sufficient to prove that every arc of M is arc-connected. Suppose there exists an arc ab which can be written as $A \cup B$ where A and B are arc-separated sets. If A is totally disconnected then Definition 3.13 implies $\overline{A} \cap B \neq \emptyset$. Thus A must contain an arc. Without loss of generality let ac be a maximal arc in A. The maximality of ac forces c to be a limit point of B. This contradicts A and B being arc-separated. Consequently, every arc of M is arc-connected and Theorem 3.27 implies M is arc-locally connected.

Let C and C' be two chains of connected open sets joining the two points a and b in the arc-topology, C having more than two links. The definition of arc-open and arc-closed sets implies the connected sets are open arcs relative to the original topology. Since M is arcwise connected and contains no simple closed curve, there exists a unique arc ab from a to b and some link of C not containing a or b intersects some link of C.

Since a Hausdorff space is T_o the mapping f is continuous in both the original and the arc-topology by Theorem 3.29. Thus, f is a continuous mapping of M into M with the arc-topology in which M is a generalized dendrite. By Theorem 3.30, f has a fixed-point.//

5. Theorem 2.8. Let D_1, D_2, D_3,... be a sequence of chains such that

i) $\overline{D}_1$ is a compact nonvacuous metric space,

ii) $\overline{D}_{i+i}$ is a subset of $\overline{D}_i$ for each i, and

iii) $\lim_{i \to \infty} \triangle(D_i) = 0$.

Let M designate the continuum which is the intersection of the $\overline{D}_i$. Then if f is a continuous transformation of M into a subset of itself, there exists a point p of M such that f(p) = p.

Proof: Let $\varepsilon > 0$. By (iii) of the hypothesis, there exists a positive integer m such that $\triangle D_m < \varepsilon$.

Let p be a point of M. Since M is the intersection of the sets $\overline{D}_i$, the point p belongs to some closed link of d_i of D_m. By the hypothesis f(p) is in M. Therefore, f(p) is in some closed link d_j of D_m where $d_j = d_i$, d_j precedes d_i, or d_i precedes d_j. Since p was arbitrary in M, this is true for every point in M.

Let A be the subset of M consisting of all points p of M such that f(p) is in a closed link of D_m following all closed links of D_m containing p or f(p) is contained in some closed link of D_m which contains p. Let B be the subset of M such that f(p) is in a closed link of D_m preceding all the closed links of D_m containing p or f(p) is contained in some link of D_m which contains p.

From Definition 2.8, the number n of links of D_m is finite. The points of M in d_i are necessarily in A and the points of M in d_n are necessarily in B. Therefore, neither A nor B is empty and $M = A \cup B$.

It can now be shown that A is closed. Suppose A is not closed. Then there exists a limit point x of A such that x is not in A. If x is not in A, then x is in $M - A = B$ and x is in some closed link d_k of D_m. Since the point x is not in A, the definitions of A and B imply that f(x) is in some closed link of D_m which precedes all the closed links of D_m containing x. The point x being a limit point of A implies that there exists a sequence of distinct points $\{a_k\}$ of A converging to x. The continuity of f implies $\{f(a_k)\}$ converges to f(x).

Let W be an open set with respect to M such that x is in W and $W \cap M$ is contained in the links of D_m which contain x. The definition of convergence implies there exists a positive integer K such that for $k > K$, a_k is in W. Therefore, for $k > K$, a_k is in the same closed links of D_m as the point x. This fact and the definition of A imply that $f(a_k)$ is in a closed link of D_m which follows all closed links of D_m containing x or $f(a_k)$ is in some closed link of D_m containing x. Thus, for $k > K$, $f(a_k)$ does not belong to the closed link of D_m containing f(x) and for some set U containing f(x), the set $U \cap f(a_k) = \emptyset$ for $k > K$. This contradicts $\{f(a_k)\}$ converging to f(x). Therefore, A is closed.

In a similar way it can be argued that B is closed. If $A \cap B = \emptyset$, then A and B are separated sets, and $M = A \cup B$ implies that M is not connected. Therefore, $A \cap B \neq \emptyset$. Let q be a point in $A \cap B$. The definitions of A and B force q and f(q) to lie together in some closed link d_r of D_m. Since $\triangle D_m < \varepsilon$, $\rho(p, f(q)) < \varepsilon$.

Because the original choice of ε was arbitrary, the preceding argument implies that for any $\varepsilon > 0$ there exists a point q of M such that $\rho(p, f(q)) < \varepsilon$. Letting $\varepsilon_n = 1/n$, it is possible to obtain a sequence $\{q_n\}$ such that $\rho(q_n, f(q_n)) < 1/n$. Since M is closed, M is compact and some subsequence of $\{q_n\}$ converges to a point of q. Without loss of generality, assume $\{q_n\}$ converges to q. The continuity of f implies $\{f(q_n)\}$ converges to f(q). Therefore, for every $\varepsilon > 0$ there exists an appropriate N such that for $n > N$, $0 \leq \rho(p, f(q)) \leq \rho(q, q_n) + \rho(q_n, f(q_n)) + \rho(f(q_n), f(q)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus $f(q) = q$.//

6. <u>Theorem 2.10.</u> Every continuous mapping of an arcwise connected and hereditarily unicoherent one-dimensional continuum into itself has a fixed-point.

<u>Proof:</u> Let A be an arcwise connected hereditarily unicoherent one-dimensional continuum. Because of the unicoherence of A and the properties of arcs, every two points a,b in A, $a \neq b$, are joined by exactly one arc ab. In case $a = b$, consider ab a degenerate arc. If p is in $ab - \{a, b\}$, the point p will be said to be in ab. Definition 3.1 and Theorem 3.1 imply a is not in pb.

Suppose there exists a continuous mapping f of A into itself without a fixed-point. Then there exists an $\varepsilon > 0$ such that $\rho(p, f(p)) > \varepsilon$ for every p in A. Inductively define a sequence $\{a_i\}$ satisfying the following conditions: i) $\rho(a_i a_{i+1}) = \varepsilon/2$ for every $i = 1, 2, \ldots$,

ii) if p is in $a_i a_{i+1}$ then $\rho(a_i, p) < \varepsilon/2$ for every $i = 1,2,\ldots,$ iii) $a_1 a_n = \bigcup_{i=1}^{n-1} a_i a_{i+1}$, and iv) a_n is in $a_1 f(a_n)$ if $n > 1$. Note that for any point a_1 these conditions are trivially satisfied.

Pick some point in A and call it a_1. Then $\rho(a_1, f(a_1)) > \varepsilon$ and there exists at least one point x in $a_1 f(a_1)$ such that $\rho(a_1, x) = \varepsilon/2$. Pick the first such point in the order from a_1 to $f(a_1)$ and call this point a_2. The point a_2 clearly satisfies condition i-iv.

Assume $a_1, a_2, \ldots a_n$ have been determined satisfying conditions i-iv. Then a_{n+1} can also be picked to satisfy these conditions. Since $\rho(a_n, f(a_n)) > \varepsilon$, there exists at least one point x in $a_n f(a_n)$ such that $\rho(x, a_n) = \varepsilon/2$. Pick the first such point in the order from a_n to $f(a_n)$ and call this point a_{n+1}. Then (i) and (ii) are satisfied.

To prove (iii) it is necessary to show that $a_1 a_n \cap a_n a_{n+1} = a_n$. Clearly $a_n \in a_1 a_n \cap a_n a_{n+1}$. Suppose there exists a point $b \neq a_n$ such that $b \in a_1 a_n \cap a_n a_{n+1}$. Then $a_1 a_{n+1} \subset (a_1 b \cup b a_{n+1}) \subset (a_1 a_n \cup a_n a_{n+1} - a_n) \subset A - a_n$.

The set $a_1 a_{n+1} \cup a_{n+1} f(a_n)$ is connected for if it weren't it could be written as $C \cup D$ sep. which leads to the conclusion that $a_1 a_{n+1}$ and $a_{n+1} f(a_n)$ are not connected. Since this is clearly a contradiction, $a_1 a_{n+1} \cup a_{n+1} f(a_n)$ is connected. Being the union of two closed and bounded sets in the plane, $a_1 a_{n+1} \cup a_{n+1} f(a_n)$ is also closed and bounded, or compact. Thus it is a continuum. Since it is a subcontinuum of an arcwise and hereditarily unicoherent one-dimensional continuum, $a_1 a_{n+1} \cup a_{n+1} f(a_n)$ is arcwise connected and possesses a unique arc $a_1 f(a_n)$. Since $a_n \notin a_{n+1} f(a_n)$, the arc $a_1 f(a_n) \subset (a_1 a_{n+1} \cup a_{n+1} f(a_n)) \subset A - a_n$. But this contradicts the induction hypothesis that a_n is in $a_1 f(a_n)$, Thus, $a_1 a_n \cap a_n a_{n+1} = a_n$ and by Theorem 3.2

$$\left(\bigcup_{i=1}^{n-1} a_i a_{i+1}\right) \cup a_n a_{n+1} = \bigcup_{i=1}^{n} a_i a_{i+1} = a_1 a_{n+1}.$$

Having proved (i), (ii) and (iii) one knows that $a_1 a_{n+1} = \bigcup_{i+1}^{n} a_i a_{i+1}$, $a_1 a_{n+1} \cap a_{n+1} a_{n+2} = a_{n+1}$, $a_{n+1} a_{n+2} \subset a_{n+1} f(a_{n+1})$ and $a_{n+1} \in a_1 a_{n+1} \cup a_{n+1} f(a_{n+1})$. Next is shown that $a_1 a_{n+1} \cap a_{n+1} f(a_{n+1}) = a_{n+1}$. Suppose the opposite and let $b \neq a_{n+1}$ be the first point in the order from a_{n+1} to $f(a_{n+1})$ such that $b \in a_1 a_{n+1} \cap a_{n+1} f(a_{n+1})$. There is a first such b since $a_1 a_{n+1} \cap a_{n+1} a_{n+2} = a_{n+1}$. Since $b \in a_1 a_{n+1}$, the set $B = \{ x/ \ x = b,\ x = a_{n+1},$ or $b < x < a_{n+1}$ on arc $a_1 a_{n+1} \}$ is the arc $b a_{n+1}$. Similarly $C = \{ x/ \ x = b,\ x = a_{n+1}$ or $a_{n+1} < b < f(a_{n+1})$ on $a_{n+1} f(a_{n+1}) \}$ is the arc $a_{n+1} b$. The choice of b implies $B \cap C = \{ a_{n+1}, b \}$ or that there are two distinct arcs joining b and a_{n+1}. This contradicts the fact that there is a unique arc joining any two points. It follows that $a_1 a_{n+1} \cap a_{n+1} f(a_n) = a_{n+1}$. As a result $a_1 a_{n+1} \cup a_{n+1} f(a_{n+1}) = a_1 f(a_{n+1})$ and a_{n+1} is in $a_1 f(a_{n+1})$ or (iv) holds for a_{n+1}.

By this process a sequence $\{a_i\}$ has been defined such that $\{a_i\}$ satisfies (i) through (iv). Let g_n be a homeomorphism mapping $[n-1,n]$ into $a_n a_{n+1}$ in such a manner that $g_n(n-1) = a_n$ and $g_n(n) = a_{n+1}$. Such a g_n exists for every n since an arc is homeomorphic to the unit interval with end points mapping onto end points.

Define $g\colon [0, \infty) \to P = \bigcup_n a_n a_{n+1}$ as follows: $g(x) = g_n(x)$ for $1 \leq x \leq n$. It will be shown that g is one-to-one, continuous and onto P. The function g is one-to-one since each $g_n(x)$ is one-to-one on $(n-1, n)$ and $g(n) = g_n(n) = g_{n+1}(n) = a_{n+1}$, $g(n-1) = g_n(n-1) = g_{n-1}(n-1) = a_n$ for every $n = 1,2,\ldots.$ The function g is onto since for any y in P, y is in $a_n a_{n+1} \cup \{a_n, a_{n+1}\}$ for some n and, consequently, there exists an x in $[n-1, n]$ such that $g_n(x) = g(x) = y$.

Let $\{x_k\}$ be a sequence of points of $[0, \infty)$ converging to x in $[0, \infty)$. If for some K it is true that x_k is in $[n-1, n]$ for all $k > K$, then $\{x_k\}$ converges to x in $[n-1, n]$ and $g(x_k) = g_n(x_k)$ converges to $g_n(x) = g(x)$. If for every K there exists a k such that x_k does not belong to a fixed $[n-1, n]$, then the definition of convergence implies there exists a K' such that for every $k > K'$, $x_k \in [n-1, n] \cup [n, n+1]$ for some fixed n and $\{x_k\} = \{y_{k_i}\} \cup \{z_{k_i}\}$ where y_{k_i} is in $[n-1, n]$ and z_{k_i} is in $[n, n+1]$ for every k_i. Applying the definition of convergence to $\{y_{k_i}\}$ and $\{z_{k_i}\}$ in this setting confirms the fact that both sequences converge to $x = n$. Consequently, $\{g_n(y_{k_i})\}$ converges to $g_n(n)$ and $\{g_{n+1}(z_{k_i})\} \to g_{n+1}(n)$ or $g(\{y_{k_i}\} \cup \{z_{k_i}\}) \to g(n) = g(x)$. Since $[0, \infty)$ is first countable, this is sufficient to show that g is continuous.

By Theorem 2.15, $\overline{P}$ is an arc. The definition of arc implies that there exists a homeomorphism h mapping $\overline{P}$ onto the interval $[0, 1]$. It will now be shown that $\{h(a_n)\}$ is monotone.

Suppose $\{h(a_n)\}$ is not monotone. Then there exists a smallest N such that $h(a_N) > h(a_{N+1})$. Since connectivity is preserved by h and the arc between any two points of each of the sets P and $[0,1]$ is unique, $h(a_{N-1}, a_N) = h(a_{N-1})h(a_N)$ is a unique arc. A similar argument implies $h(a_N)h(a_{N+1})$ is a unique arc. Since $h(a_{N-1})$ precedes $h(a_N)$ as does $h(a_{N+1})$, and $h(a_N) \neq h(a_{N+1})$, the set $h(a_N)h(a_{N+1}) \cap h(a_{N-1})h(a_N) - h(a_N) \neq \emptyset$; or h is not one-to-one. This contradicts the fact that h is a homeomorphism. Consequently, $\{h(a_n)\}$ is monotone.

Since any bounded monotone sequence of real numbers is convergent, $\{h(a_n)\}$ is convergent. The continuity of h^{-1} implies $\{a_n\}$ is also convergent. But this contradicts the fact that $\rho(a_n, a_{n+1}) = \epsilon/2$ for every $n = 1,2,\ldots$ Thus the original assumption was false; that is, f has a fixed-point.

7. Theorem 2.11. If X is an arcwise connected Hausdroff space which contains no circle, and if there exists an e in X such that K_r has the fixed point property for each e-ray R, then X has the fixed-point property.

Proof: Let f be a continuous function, $f\colon X \to X$. If $f(e) = e$, the proof is complete. Assume $f(e) \neq e$. Let D denote the family of all subsets S of X such that $e \in S$, $S \cup f(S)$ is linearly ordered with respect to the partial order of Definition 3.2 and $s \leq f(s)$ for each s in S.

Clearly $\{e\}$ is a subset of X. By Definition 3.2, the set $\{e, f(e)\}$ is linearly ordered with $e \leq f(e)$. Consequently, $\{e\}$ is in D and $D \neq \emptyset$. Partially order D by set containment. Let C be a chain in D. Let $K = \bigcup_{S \in C} S$. Clearly $K \subset X$. For every s in K, $s \in S$ for some S in D and, consequently, $s \leq f(s)$. To show that $K \cup f(K)$ is linearly ordered with respect to $\leq$, consider the following cases: i) s and t are in K, ii) s and t are in f(K), and iii) $s \in K$ and $t \in f(K)$.

If s and t are in K, then there exist sets S and T in C such that $s \in S$ and $t \in T$. Since C is a chain assume, without loss of generality, that $S \subset T$. The linear order of T implies $s \leq t$ or $t \leq s$.

If s and t are in f(K) then there exist x and y in S and T, respectively, such that $f(x) = s$, $f(y) = t$. Because S and T are in C, assume without loss of generality that $S \subset T$. This implies $f(S) \subset f(T)$. The linear order of $T \cup f(T)$ implies $s \leq t$ or $t \leq s$.

If $s \in K$, $t \in f(K)$, then there exists a $y \in T$ in C such that $f(y) = t$. If $S \subset T$, the linear order of T implies $s \leq f(y) = t$ or $t = f(y) \leq s$. If $T \subset S$, then $f(T) \subset f(S)$ and the linear order of $S \cup f(S)$ implies the same result.

The preceding argument shows that $K \cup f(K) \in C$. Since $S \subset K$, the set K is an upper bound for S. By Zorn's Lemma, D has a maximal element. Call it M.

Suppose $M \cup f(M) \subset ex$ for some $x \in X$. If $x \nleq f(x)$ then for some $m \in M$, $m \leq f(m) \leq x$ and f must have a fixed-point by Theorem 3.14. In this case, the proof is complete.

Suppose $x \leq f(x)$ for each x such that $M \cup f(M) \subset ex$. Since M is maximal and $x \leq f(x)$, $x \in M$. Since $M \cup f(M) \subset ex \subset ef(x)$, the assumption implies $f(x) \leq f(f(x))$. The maximality of M implies $f(x) \in M$. This means $f(x) \in ex$ or $f(x) \leq x$. Since $x \leq f(x)$ and $f(x) \leq x$, $x = f(x)$.

If $M \cup f(M) \not\subset ex$ for any $x \in X$ then for some e-ray R the following is true: for each $r \in R$ there exists an element m called $m(r)$, $m(r) \in M \cup f(M)$ such that $r \leq m(r)$. In other words, $M \cup f(M)$ is cofinal in R. Moreover, M is cofinal in R. Suppose this is not true. Then there exists an $r \in R$ such that for every $m \in M$, $m \leq r$. This means $M \cup f(M) \subset er \cup f(er)$. Since $e \leq f(e)$, $f(e) \in R$. Since X contains no circle and $f(er)$ is arcwise connected, $f(er) \cap R$ is the point $f(e)$ or an arc $f(e)b$ for some b in T. Since $f(M) \subset R$, $f(M) = f(e)$ or $f(M) \subset f(e)b$. If $f(M) = f(e)$ then for every $m \in M \cup f(M)$, $m \leq f(e)$ which contradicts $M \cup f(M)$ cofinal in R. If $f(M) \subset f(e)b$ either $f(e) \leq b$ or $b \leq f(e)$, it follows that for every $m \in M \cup f(M)$, $m \leq b$ or $m \leq f(e)$. This contradicts the statement that $M \cup f(M)$ is cofinal in

R. In any case M is confinal in R.

If $f(K_r) \subset K_r$ for every K_r, the proof is complete. Assume $f(K_r) - K_r \neq \emptyset$. This implies $K_r \neq \emptyset$. Let $y \in K_r$ such that $f(y) \in X - K_r$. Define g: $R \to R$ such that $g(x) = x$ and note that the linear order $\geq$ on R is such that $(R, \geq)$ is a directed set. Thus, the net $(g, \geq)$ exists. Let U be an open set such that $y \in U$. For each x in R there exists $p \geq x$ such that $g(p) = p \in U$. Clearly U - ex is open. Since y is a limit point of R, the set $(U - ex) \cap R \neq \emptyset$. It is possible to pick a $p \in (U - ex) \cap R$, $p \geq x$ and $p = g(p) \in U$. Thus, by Kelley (28) page 71, y is a cluster point of the net $(g, \geq)$, and some subnet converges to y. Name it $(g_E, \geq)$ and choose E so that $E \subset R$.

Suppose there exists an $x \in E$ such that $x \notin ef(x)$. Since M is cofinal in $R \supset E$, there exists an $m \in M$ such that $m \geq x$ and $m \leq f(m)$. Hence, $x \leq m \leq f(m)$.

Since X contains no circle and the arc joining any two points of X is unique, $x < m$ in the usual order from f(x) to f(m) on the arc f(x)f(m). By Theorem 3.5 f has a fixed-point.

Suppose $x \in ef(x)$ for every x in E. Since $f(y) \notin K_r$, f is continuous, and $(g_E, \geq)$ converges to y, it may be assumed without loss of generality that $f(x) \notin K_r$ for every x in E. Suppose there exists an x in E such that $f(x) \leq f(f(x))$. Since $f(x) \notin R$ and $f(x) \leq f(f(x))$ there exists an $m \in M$ such that $m \notin ef(f(x))$. If this were not true, the e-ray R would be a proper subset of ef(f(x)). This contradicts the definition of R.

Since $m \leq f(m)$, the arc $mf(m) \subset R$ has no point in common with $ef(f(x)) \supset f(x)f(f(x))$. It follows from elementary arc properties and the fact that X contains no circle that $m < f(x)$ in the natural order from $f(m)$ to $f(f(x))$ on the arc $f(m)f(f(x))$. By Theorem 3.5 f has a fixed-point.

The only remaining alternative is $x \leq f(x) \not\leq f(f(x))$. In this case the hypothesis of Theorem 3.4 is satisfied, $x \leq f(x) \leq f(x)$, and f has a fixed-point.

Accordingly, either $f(K_r) \subset K_r$ for every R or f has a fixed-point. If $f(K_r) \subset K_r$, f has a fixed-point by the hypothesis. Consequently, X has the fixed-point property.//

8. Theorem 2.18. If H is a hereditarily unicoherent continuum such that $\tau(H) \neq \infty$, then H has the fixed-point property.

Proof: Suppose the opposite. Then by Theorem 3.33, H contains a subcontinuum H' with $\tau(H') \neq \infty$ and such that $f_{H'}$ maps H' onto H'. Consequently, it is sufficient to consider H a specific hereditarily unicoherent continuum for which $\tau(H) = h$ and f: $H \to H$ is an onto map.

Let x denote a particular point in $J^h(H)$ and let A denote the set of all points in H which can be joined to x by an arc in H. Then A is arcwise connected by elementary arc properties. Using Young's Theorem 2.7, it will suffice to show that every monotone increasing sequence of arcs in A is contained in an arc in A.

For given a,b in A, $ab = I\{a,b\}$ in H by Moore (35) page 40. By Theorem 2.16, the arc ab is unique. Let $\{a_n b_n\}$ be a sequence of monotone increasing arcs in A and let $p \in \bigcup_n a_n b_n$, $p \neq a_n$, $p \neq b_n$ for some n, then by Theorem 3.34 it can be assumed that $\overline{\bigcup pb_n}$ is an arc. That is, there exists a $y \in \overline{\bigcup a_n b_n}$ such that $py = \overline{\bigcup_n pb_n}$. By Theorem 3.35, $(\overline{\bigcup_n a_n p}) \cup (\overline{\bigcup_n pb_n}) = (\overline{\bigcup_n a_n p}) \cup (py) = \overline{\bigcup_n a_n y} \subset A$. If it can be shown that $\overline{\bigcup_n a_n y}$ is an arc, the proof of Theorem 2.18 will be complete.

Suppose $\overline{\bigcup_n a_n y}$ is not an arc. Then by Theorem 3.36 $\overline{\bigcup_n a_n y}$ fails to be locally connected at every point of the set $L(y) = \overline{\bigcup_n a_n y} - \bigcup_n a_n y$. Since $\tau(H) = h \neq \infty$ there exists a $\beta = \sup \{ \gamma / L(y) \subset J^{\gamma}(H) \}$. Thus $L(y) \subset J^{\beta}(H)$ and $L(y) \not\subset J^{\beta+1}(H)$. The definition of J^{β} implies $J^{\beta}(H)$ is locally connected at some $z \in L(y)$.

It will now be shown that $J^{\beta}(H) \cap (\bigcup_n a_n y) = \emptyset$. Suppose there exists a $q \in J^{\beta}(H) \cap (\bigcup_n a_n y)$. Since $J^{\beta}(H)$ is a continuum, $I(\{z, q\}) \subset J^{\beta}(H)$. By Theorem 3.37 $I(\{z,q\}) = \overline{\bigcup_n a_n q}$. By Theorem 3.35 $(\overline{\bigcup_n a_n q}) \cup (qy) = \overline{\bigcup_n a_n y}$ and $\overline{\bigcup_n a_n q} \cap qy = q$. Since $\overline{\bigcup_n a_n y}$ is not locally connected at z, these relations imply $\overline{\bigcup_n a_n q}$ is not locally connected at z. Since $\overline{\bigcup_n a_n q} = I(\{z,q\}) \subset J^{\beta}(H)$, Theorem 3.35 implies $N(\overline{\bigcup_n a_n q}) \subset N(J^{\beta}(H))$ or $J^{\beta}(H)$ is not locally connected at z. This contradiction concludes the proof that $J^{\beta}(H) \cap (\bigcup_n a_n y) = \emptyset$.

Let xy be the unique arc joining y to the point x by which A was determined. Theorem 3.32 implies $J^{h}(H) \subset J^{\beta}(H)$. Therefore, $x \in J^{\beta}(H)$. Thus $\{x,y\} \subset J^{\beta}(H) \cup (\overline{\bigcup_n a_n y})$ which is connected since both $J^{\beta}(H)$ and $\overline{\bigcup_n a_n y}$ are connected with $z \in J^{\beta}(H) \cap (\overline{\bigcup_n a_n y})$. It follows that $xy = I(\{x,y\}) \subset J^{\beta}(H) \cup (\overline{\bigcup_n a_n y})$.

It will now be shown that $xy \cap (\overline{\bigcup a_n y}) = \overline{\bigcup_n a_n y}$ or, $\overline{\bigcup_n a_n y}$ is indeed an arc. By definition $\overline{\bigcup_n a_n y} \subset xy$. Suppose $p \in xy$ and $p \notin xy \cap (\overline{\bigcup_n a_n y})$. By Theorem 3.35, $\overline{\bigcup_n a_n y} = \overline{\bigcup_n a_n p} \cup py$ and $\overline{\bigcup_n a_n p} \cap py = p$. Because H is hereditarily unicoherent, $\overline{\bigcup_n a_n y} \cap xy$ is a continuum. Since $p \notin \overline{\bigcup_n a_n y} \cap xy$, the preceding relationships infer that $\overline{\bigcup_n a_n y} \cap xy \subset py$.

Because $xy \subset J^{\beta}(H) \cup (\overline{\bigcup a_n y})$ and $\overline{\bigcup_n a_n y} \cap xy \subset py$, $xy \subset py \cup J^{\beta}(H)$ sep. The sets py and $J^{\beta}(H)$ are separated because $py \cap J^{\beta}(H) = \emptyset$

and both py and $J\beta(H)$ are closed. Since xy is connected and $y \in xy \cap py$, $xy \subset py$ and $xy \cap J\beta(H) = \emptyset$. This contradicts the original choice of x in $J^h(H) \subset J\beta(H)$ and proves that $p \in xy \cap (\overline{\cup a_n y})$. But this means $py \subset \overline{\cup a_n y} \cap xy$ for any $p \in \cup a_n y$. Consequently, $\cup a_n y \subset \overline{\cup a_n y} \subset \overline{(\cup a_n y) \cup xy} = \overline{\cup a_n y} \cup \overline{xy} = \overline{\cup a_n y} \cup xy$ and $\overline{\cup a_n y}$ is an arc.

Thus, any monotone increasing sequence of arcs in A is contained in an arc, and by Theorem 2.7 $A \subset H$ has a fixed-point.//

APPENDIX B

INDEX OF TERMS

VITA

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Doctor of Education

Thesis: THE FIXED-POINT QUESTION FOR BOUNDED NON-SEPARATING PLANE CONTINUA

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